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2-unitary operads of GK-dimension 3

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This paper is dedicated to the memory of Earl J. Taft.

Abstract. We study and classify the 2-unitary operads of Gelfand–Kirillov (GK) dimension 3.

1. Introduction

Algebraic operads originated from homotopy theory in algebraic topology, and was first introduced by Boardman–Vogt [**BV**] and May [**Ma**] in the 1960s and 1970s. Throughout the past twenty years, operad theory has become an important tool in homological algebra, category theory, algebraic geometry, and mathematical physics. It is well known that every operad encodes an algebra system. For example, $\mathcal{A}ss$ encodes all unital associative algebras. Further, a $\Bbbk$-linear operad itself is an algebraic object similar to an associative algebra, and algebraic structures of operads have been widely investigated by many mathematicians; see [**BYZ**, **Dot**, **DK**, **DMR**, **DT**, **Fr1**, **Fr2**, **KP**, **LV**, **MSS**, **QXZZ**].

The Gelfand–Kirillov dimension of an associative algebra is a useful numerical invariant in ring theory and noncommutative algebraic geometry; see [**KL**]. In a similar way, the Gelfand–Kirillov dimension can be defined for other algebraic objects, including algebraic operads [**BYZ**, **Fi**]. Let $\Bbbk$ be a base field. An operad $\mathcal{P}$ is said to be *locally finite* if each $\mathcal{P}(n)$ is finite dimensional over $\Bbbk$. In this paper we only consider locally finite operads. The *Gelfand–Kirillov dimension* (or *GK-dimension* for short) of a locally finite operad $\mathcal{P}$ is defined to be

$$\operatorname{GKdim} \mathcal{P} \colon = \limsup_{n\to\infty} \log_n \left(\sum_{i=0}^{n} \dim_{\Bbbk} \mathcal{P}(i) \right).$$

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We refer to [**BYZ**, **KP**, **QXZZ**] for more information related to the GK-dimension of an operad.

Recall that an operad $\mathcal{P}$ is *unitary* if $\mathcal{P}(0) = \Bbbk \mathbb{1}_0$ with a basis element $\mathbb{1}_0$ (which is called a 0-unit), see [**Fr2**, Section 2.2]. Denote by Op_+ the category of unitary operads, in which a morphism preserves the 0-unit. A *2-unitary operad* $\mathcal{P}$ is a unitary operad $\mathcal{P}$ equipped with a morphism $\mathcal{M}ag \to \mathcal{P}$ in Op_+, where $\mathcal{M}ag$ is the unital magmatic operad (see [**BYZ**, Section 8.4] or [**Lo**, Section 4.1.10]). The definition of a $2a$-unitary operad is given in Definition 2.5(4). In [**BYZ**], the authors proved that GK-dimension of a 2-unitary operad $\mathcal{P}$ is either an nonnegative integer or infinity and that the generating series of $\mathcal{P}$ is a rational function when $\operatorname{GKdim} \mathcal{P} < \infty$. The pattern of GK-dimension of a general non-2-unitary operad (or nonsymmetric operad) is very different; see Remark 7.5.

The only 2-unitary operad of GK-dimension 1 is $\mathcal{C}om$, which encodes all unital commutative algebras. All locally finite 2-unitary operads of GK-dimension 2 were classified in [**BYZ**, Theorem 0.6]. One way of viewing this classification is the following. We refer to [**BYZ**, Section 6] for the construction of 2-unitary operads of GK-dimension 2.

Theorem 1.1 ([**BYZ**, Theorem 0.6]). *There are natural equivalences between*

1. *the category of finite dimensional, not necessarily unital, $\Bbbk$-algebras,*
2. *the category of 2-unitary operads of GK-dimension ≤ 2,*
3. *the category of $2a$-unitary operads of GK-dimension ≤ 2.*

In the case of GK-dimension 3, our result is not as clean as Theorem 1.2. Nevertheless, we will provide a classification. Recall that an operad is called *$\mathcal{C}om$-augmented* if there is an operadic unit map $u_{\mathcal{P}} : \mathcal{C}om \to \mathcal{P}$. The morphisms in the category of $\mathcal{C}om$-augmented operads are supposed to be compatible with operadic unit maps. An algebra over a $\mathcal{C}om$-augmented operad is a commutative algebra together with additional operations on it. A typical example of $\mathcal{C}om$-augmented operad is the Poisson operad $\mathcal{P}ois$. Here is the main result of this paper.

Theorem 1.2. *There is a natural equivalence between*

1. *the category of finite dimensional trident algebras,*
2. *the category of $\mathcal{C}om$-augmented operads of GK-dimension ≤ 3.*

If $\operatorname{char} \Bbbk \neq 2$, we also prove that every 2-unitary operad of GK-dimension 3 is equipped with a $\mathcal{C}om$-augmentation with possibly new 2-unit (Proposition 3.5). Combining Theorem 1.2 with Proposition 3.5, we obtain a classification of 2-unitary operads of GK-dimension 3 in terms of finite dimensional trident algebras. However, Theorem 1.2 fails if the condition "$\mathcal{C}om$-augmented" in part (2) is replaced by "2-unitary" (Example 7.4).

The definition of a trident algebra is given in Section 4. Roughly speaking, a trident algebra consists of a pair of $\Bbbk$-vector spaces (A, M) equipped with some algebraic structures and a pair of $\Bbbk$-linear maps (f, g) satisfying some identities. One special case is when $M = 0$, then both f and g must be zero maps. The corresponding operads have GK-dimension 2, which is in the situation of Theorem 1.1. Another special case is when $A = \Bbbk$ and M is a nonzero finite dimensional right $\mathbb{S}_2$-module (with both f and g being 0 automatically). Then $(\Bbbk, M)$ is a trident algebra. The trident algebras of the form $(\Bbbk, M)$ correspond exactly to connected 2-unitary (and $\mathcal{C}om$-augmented) operads of GK-dimension three. So we have the following corollary.

Corollary 1.3 (Corollary 6.1). *The category of connected 2-unitary operads of GK-dimension 3 is equivalent to the category of nonzero right $\mathbb{S}_2$-modules.*

Note that a seemingly technical result, Theorem 1.2, makes some questions easy to solve. For example, we prove Corollary 1.4.

Corollary 1.4 (Proposition 7.3). *There is no $\mathcal{C}om$-augmented Hopf operad of GK-dimension 2.*

Corollary 1.4 motivates the following question.

Question 1.5. Is there a $\mathcal{C}om$-augmented Hopf operad of finite GK-dimension larger than 2?

This paper is organized as follows. In Section 2 we recall some basic definitions and properties of 2-unitary operads. We prove some properties of 2-unitary operads of GK-dimension 3 in Section 3. A key preliminary result is that a 2-unitary operad of GK-dimension 3 is $\mathcal{C}om$-augmented after changing the 2-unit (Proposition 3.5). We define a concept of a trident algebra in Section 4. In Section 5, we construct an operad from a trident algebra and complete the classification of 2-unitary operads of GK-dimension 3 by Theorem 1.2 and Proposition 3.5. In Section 6, we describe all connected 2-unitary operads of GK-dimension three. In Section 7, we give some comments, examples, and remarks.

2. Preliminaries

Throughout, let $\Bbbk$ be a base field, and every object is over $\Bbbk$. Let n be a nonnegative integer. Set $[n] = \{1, 2, \cdots, n\}$ for $n > 0$ and $[0] = \varnothing$. We use $\mathbb{S}_n$ to denote the symmetric group for $n \geq 0$. By convention, $\mathbb{S}_0$ is the trivial group. Following the notation introduced in [**BYZ**, Section 8.1], for each $\sigma \in \mathbb{S}_n$, we use the sequence $(i_1, i_2, \cdots, i_n)$ to denote a permutation $\sigma \in \mathbb{S}_n$ with $\sigma(i_k) = k$ for all $1 \leq k \leq n$. Denote by $\mathbb{S}$ the disjoint union of all symmetric group $\mathbb{S}_n$ for all $n \geq 0$. Recall that a $\Bbbk\mathbb{S}$-module (or $\mathbb{S}$-module) means a sequence $\{\mathcal{P}(n)\}_{n \geq 0}$ of right $\Bbbk\mathbb{S}_n$-modules, where the right $\mathbb{S}_n$-action on $\mathcal{P}(n)$ is denoted by $*$.

In this section, we retrospect some basic facts about operads.

2.1. Definitions. From different viewpoints, there are various equivalent definitions about operads. In this paper, we mainly use the *partial* definition and refer to [**LV**, Chapter 5] for other versions of the definition.

Definition 2.1 ([**Fr2**, Section 2.1]). An *operad* $\mathcal{P}$ consists of the following data:

(i) a $\Bbbk\mathbb{S}$-module $\{\mathcal{P}(n)\}_{n \geq 0}$, where an element in $\mathcal{P}(n)$ is called an *n-ary operation*.

(ii) an element $\mathbb{1} \in \mathcal{P}(1)$, which is called the *identity*,

(iii) for all $m \geq 1, n \geq 0$ and $1 \leq i \leq m$, a *partial composition*:

$$- \underset{i}{\circ} -\colon \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m + n - 1)$$

satisfying the following coherence axioms:

(OP1) (Identity) for all $\theta \in \mathcal{P}(m)$ and all $1 \leq i \leq m$,

$$\theta \underset{i}{\circ} \mathbb{1} = \theta = \mathbb{1} \underset{1}{\circ} \theta;$$

(OP2) (Associativity) for all $\lambda \in \mathcal{P}(l), \mu \in \mathcal{P}(m)$ and $\nu \in \mathcal{P}(n)$,

(E2.1.1) $$(\lambda \underset{i}{\circ} \mu) \underset{i-1+j}{\circ} \nu = \lambda \underset{i}{\circ} (\mu \underset{j}{\circ} \nu), \quad 1 \leq i \leq l, 1 \leq j \leq m,$$

(E2.1.2) $$(\lambda \underset{i}{\circ} \mu) \underset{k-1+m}{\circ} \nu = (\lambda \underset{k}{\circ} \nu) \underset{i}{\circ} \mu, \quad 1 \leq i < k \leq l;$$

(OP3) (Equivariance) for all $\mu \in \mathcal{P}(m), \nu \in \mathcal{P}(n)$ and $\sigma \in \mathbb{S}_n$, $\phi \in \mathbb{S}_m$,

(E2.1.3) $$\mu \underset{i}{\circ} (\nu * \sigma) = (\mu \underset{i}{\circ} \nu) * \sigma',$$

(E2.1.4) $$(\mu * \phi) \underset{i}{\circ} \nu = (\mu \underset{\phi(i)}{\circ} \nu) * \phi'',$$

where $\sigma' = 1_m \underset{i}{\circ} \sigma$ and $\phi'' = \phi \underset{i}{\circ} 1_n$ are given by the partial composition in the associative algebra operad $\mathcal{A}ss$. We refer to [**BYZ**, Section 8] for more details concerning σ' and ϕ''.

Let $\mathcal{P}$ be an operad in the sense of Definition 2.1. Then one can define the composition map by

(E2.1.5) $$\lambda \circ (\mu_1, \cdots, \mu_n) = (\cdots((\lambda \underset{n}{\circ} \mu_n) \underset{n-1}{\circ} \mu_{n-1}) \underset{n-2}{\circ} \mu_{n-2} \cdots) \underset{1}{\circ} \mu_1$$

for all $\lambda \in \mathcal{P}(n)$ and $\mu_i \in \mathcal{P}$ and for $1 \leq i \leq n$ [**BYZ**, Remark 1.3].

Example 2.2 ([**LV**, Section 5.2.10]). Let $\mathcal{C}om$ denote the commutative algebra operad. The space of n-ary operations of $\mathcal{C}om$ is $\mathcal{C}om(n) = \Bbbk \mathbb{1}_n$ equipped with the trivial action of the symmetric group and the partial composition is given by $\mathbb{1}_m \underset{i}{\circ} \mathbb{1}_n = \mathbb{1}_{m+n-1}$ for all m, n, i. Note that $\mathbb{1}_1$ is the identity $\mathbb{1}$ of $\mathcal{C}om$.

Example 2.3 ([**LV**, Section 13.8]). Suppose that $\mathcal{M}ag$ is the operad generated by the $\mathbb{S}$-module

$$(\Bbbk\mu, \Bbbk\mathbb{1}, \Bbbk\mathbb{S}_2\nu, 0, 0, \cdots)$$

and subject to relations

$$\nu \underset{i}{\circ} \mu = \mathbb{1}, (i = 1, 2),$$

where $\Bbbk\mathbb{S}_2\nu$ is the regular $\Bbbk\mathbb{S}_2$-module with the basis ν. In this paper we use $\mathbb{1}_0$ for μ and $\mathbb{1}_2$ for ν.

Definition 2.4 ([**LV**, Chapter 5]). Let $\mathcal{P}$ and $\mathcal{P}'$ be operads. A *morphism* from $\mathcal{P}$ to $\mathcal{P}'$ is a sequence of $\Bbbk\mathbb{S}_n$-homomorphism $\gamma = (\gamma_n : \mathcal{P}(n) \to \mathcal{P}'(n))_{n \geq 0}$, satisfying

$$\gamma(\mathbb{1}) = \mathbb{1}' \quad \text{and} \quad \gamma(\mu \underset{i}{\circ} \nu) = \gamma(\mu) \underset{i}{\circ} \gamma(\nu),$$

where $\mathbb{1}$ and $\mathbb{1}'$ are identities of $\mathcal{P}$ and $\mathcal{P}'$, respectively, and $\mu, \nu \in \mathcal{P}$. Let Op denote the category of operads.

Next we collect some definitions given in [**BYZ**, **Fr2**].

Definition 2.5.

(1) An operad $\mathcal{P}$ is called *unitary* if $\mathcal{P}(0) = \Bbbk\mathbb{1}_0$, where $\mathbb{1}_0$ is a basis of $\mathcal{P}(0)$, and is called a 0-*unit*. The category of unitary operads is denoted by Op_+, in which morphisms are operadic morphisms preserving 0-units.

(2) A unitary operad is said to be *connected*, if $\mathcal{P}(1) = \Bbbk\mathbb{1}$ where $\mathbb{1}$ is the identity of $\mathcal{P}$. In this case we also use $\mathbb{1}_1$ for $\mathbb{1}$.

(3) A *2-unitary operad* is a unitary operad $\mathcal{P}$ equipped with a morphism $\mathcal{M}ag \to \mathcal{P}$ in Op_+, where $\mathcal{M}ag$ is the unital magmatic operad [Example 2.3], or equivalently, there is an element $\mathbb{1}_2 \in \mathcal{P}(2)$ (called a 2-unit) such that

$$\mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_0 = \mathbb{1}(= \mathbb{1}_1) = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_0 \tag{E2.5.1}$$

where $\mathbb{1}_0$ is a 0-unit of $\mathcal{P}$.

(4) A *2a-unitary* operad is a unitary $\mathcal{P}$ equipped with a morphism $\mathcal{A}ss \to \mathcal{P}$ in Op_+, or equivalently, $\mathcal{P}$ is 2-unitary with a 2-unit $\mathbb{1}_2$ satisfying

$$\mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_2. \tag{E2.5.2}$$

In this case $\mathbb{1}_2$ is called $2a$*-unit.*

(5) An operad $\mathcal{P}$ is called $\mathcal{C}om$*-augmented* if there is a morphism from $\mathcal{C}om \to \mathcal{P}$. It is clear that $\mathcal{C}om$-augmented operads are $2a$-unitary. In this case $\mathbb{1}_2 * (2,1) = \mathbb{1}_2$ and $\mathbb{1}_2$ is called a *symmetric* $2a$*-unit.*

Let dim denote $\dim_{\Bbbk}$.

Definition 2.6. Let $\mathcal{P} = (\mathcal{P}(n))_{n\geq 0}$ be a locally finite operad, i.e. $\dim \mathcal{P}(n) < \infty$ for all $n \geq 0$.

(1) The *generating series* of $\mathcal{P}$ is defined to be

$$G_{\mathcal{P}}(t) = \sum_{n=0}^{\infty} \dim \mathcal{P}(n) t^n \in \mathbb{Z}[[t]].$$

(2) The *Gelfand-Kirillov dimension* (*GK-dimension* for short) of $\mathcal{P}$ is defined to be

$$\mathrm{GKdim}(\mathcal{P}) = \limsup_{n\to\infty} \log_n \Big(\sum_{i=0}^{n} \dim \mathcal{P}(i)\Big).$$

Example 2.7.

(1) The Poisson operad $\mathcal{P}ois$ is $\mathcal{C}om$-augmented. In fact, the operad $\mathcal{P}ois$ fits into the sequences $\mathcal{C}om \hookrightarrow \mathcal{P}ois \twoheadrightarrow \mathcal{L}ie$ and $\mathcal{L}ie \hookrightarrow \mathcal{P}ois \twoheadrightarrow \mathcal{C}om$, where $\mathcal{L}ie$ is the Lie operad, see for instance [**LV**, 13.3.3]. Note that the Lie operad $\mathcal{L}ie$ is not $\mathcal{C}om$-augmented.

(2) A 2-unitary operad of GK-dimension 2 is always $\mathcal{C}om$-augmented, see [**BYZ**, Lemma 6.2]. Moreover, as we will show below, if $\mathrm{char}\,\Bbbk \neq 2$, then every 2-unitary operad of GK-dimension 3 is equipped with a $\mathcal{C}om$-augmentation with possibly new 2-unit [Proposition 3.5].

Let $\mathcal{P}$ and $\mathcal{P}'$ be 2-unitary operads. A morphism of 2-unitary operads is a morphism $\gamma : \mathcal{P} \to \mathcal{P}'$ in Op_+ satisfying the following commutative diagram

$$\begin{tikzcd} & \mathcal{M}ag \arrow[dl] \arrow[dr] & \\ \mathcal{P} \arrow[rr, "\gamma"] & & \mathcal{P}' \end{tikzcd}$$

or equivalently, the operad morphism preserves the 2-unit. The categories of 2-unitary operads, 2a-unitary operads and $\mathcal{C}om$-augmented operads, are denoted by $\mathcal{M}ag \downarrow \mathrm{Op}_+$, $\mathcal{A}ss \downarrow \mathrm{Op}_+$ and $\mathcal{C}om \downarrow \mathrm{Op}_+$, respectively.

2.2. Truncation ideals. Let $\mathcal{P}$ be a unitary operad and I a subset of $[n]$. Recall that a *restriction operator* [**Fr2**, Section 2.2.1] means

$$\pi^I\colon \mathcal{P}(n) \to \mathcal{P}(s), \qquad \pi^I(\theta) = \theta \circ (\mathbb{1}_{\chi_I(1)}, \cdots, \mathbb{1}_{\chi_I(n)}) \tag{E2.7.1}$$

for all $\theta \in \mathcal{P}(n)$, where χ_I is the characteristic function of I, i.e. $\chi_I(x) = 1$ for $x \in I$ and $\chi_I(x) = 0$ otherwise. Note that $\circ$ is defined in (E2.1.5). If $I = \{i_1, \cdots, i_s\} \subset [n]$ with $1 \le i_1 < \cdots < i_s \le n$, we also denote π^I as $\pi^{i_1,\cdots,i_s}$. We refer to [**BYZ**, Section 2.3] for more details.

For integers $k \ge 1$, the kth truncation ideals ${}^k\Upsilon$ is defined by

$$ {}^k\Upsilon_{\mathcal{P}}(n) = \begin{cases} \bigcap\limits_{I\subset[n],\, |I|=k-1} \ker \pi^I, & \text{if } n \ge k; \\ 0, & \text{otherwise.} \end{cases} \tag{E2.7.2}$$

By convention, let ${}^0\Upsilon_{\mathcal{P}} = \mathcal{P}$. If no confusion, we write ${}^k\Upsilon = {}^k\Upsilon_{\mathcal{P}}$ for brevity.

For every subset $I = \{i_1, \cdots, i_s\} \subset [n]$ with $i_1 < \cdots < i_s$, we denote a permutation

$$c_I\colon = (i_1, \cdots, i_s, 1, \cdots, i_1 - 1, i_1 + 1, \cdots, i_s - 1, i_s + 1, \cdots, n) \in \mathbb{S}_n.$$

THEOREM 2.8 ([**BYZ**, Theorem 4.6]). *Let $\mathcal{P}$ be a 2-unitary operad. For each $k \ge 0$, let*

$$\Theta^k\colon = \{\theta_1^k, \cdots, \theta_{z_k}^k\}$$

be a $\Bbbk$-basis for ${}^k\Upsilon(k)$. Let $\mathbf{B}_k(n)$ be the set

$$\{\mathbb{1}_2 \circ (\theta_i^k, \mathbb{1}_{n-k}) * c_I \mid 1 \le i \le z_k, I \subset [n], |I| = k\}.$$

Then $\mathcal{P}(n)$ has a $\Bbbk$-basis

$$\bigcup_{0\le k\le n} \mathbf{B}_k(n) = \{\mathbb{1}_n\} \cup \bigcup_{1\le k\le n} \mathbf{B}_k(n),$$

and for every $k \ge 1$, ${}^k\Upsilon(n)$ has a $\Bbbk$-basis $\bigcup_{k\le i\le n} \mathbf{B}_i(n)$.

LEMMA 2.9 ([**BYZ**, Lemma 5.2]). *Let $\mathcal{P}$ be a 2-unitary operad and $f_{\mathcal{P}}(k) = \dim {}^k\Upsilon(k)$ for each $k \ge 0$. Then*

(1) $G_{\mathcal{P}}(t) = \sum\limits_{k=0}^{\infty} f_{\mathcal{P}}(k) \dfrac{t^k}{(1-t)^{k+1}}$.

(2) $\operatorname{GKdim} \mathcal{P} = \max\{k \mid f_{\mathcal{P}}(k) \ne 0\} + 1 = \min\{k \mid {}^k\Upsilon = 0\}$.

Combining Lemma 2.9(2) with [**BYZ**, Proposition 0.5], if $\mathcal{P}$ is 2-unitary, then there is a canonical morphism of 2-unitary operads

$$\epsilon: \quad \mathcal{P} \longrightarrow \mathcal{P}/{}^1\Upsilon = \mathcal{C}om. \tag{E2.9.1}$$

3. Basic facts on 2-unitary operads of GK-dimension 3

Let $\mathcal{P}$ be a 2-unitary operad of GK-dimension 3. By Theorem 2.8 and Lemma 2.9, we have the following basic facts:

(1) ${}^2\Upsilon \ne 0$, and ${}^k\Upsilon = 0$ for all $k \ge 3$.

(2) $\dim \mathcal{P}(n) = 1 + dn + m\dfrac{n(n-1)}{2}$, where $d := f_{\mathcal{P}}(1) = \dim {}^1\Upsilon(1)$ and $m := f_{\mathcal{P}}(2) = \dim {}^2\Upsilon(2)$.

(3) $G_{\mathcal{P}}(t) = \dfrac{1}{1-t} + d\dfrac{t}{(1-t)^2} + m\dfrac{t^2}{(1-t)^3}$.

Based on the above facts, we have the following lemmas, which is useful to understand the structure of a 2-unitary operad of GK-dimension 3.

LEMMA 3.1. *Let $\mathcal{P}$ be a 2-unitary operad of GK-dimension 3 and $\mathbb{1}_2$ a 2-unit. Then for all $\tau, \mu \in {}^2\Upsilon(2)$ and $\delta \in {}^1\Upsilon(1)$,*

$$\tau \underset{1}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{2}{\circ} \tau + (\mathbb{1}_2 \underset{2}{\circ} \tau) * (2,1,3), \tag{E3.1.1}$$

$$\tau \underset{2}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{1}{\circ} \tau + (\mathbb{1}_2 \underset{1}{\circ} \tau) * (1,3,2). \tag{E3.1.2}$$

$$\tau \underset{i}{\circ} \mu = 0 \quad (i = 1, 2). \tag{E3.1.3}$$

$$\mathbb{1}_2 \circ (\mu, \delta) = 0, \tag{E3.1.4}$$

$$\mathbb{1}_2 \circ (\tau, \mu) = 0, \tag{E3.1.5}$$

PROOF. By a direct calculation, we have

$$(\tau \underset{1}{\circ} \mathbb{1}_2 - \mathbb{1}_2 \underset{2}{\circ} \tau - (\mathbb{1}_2 \underset{2}{\circ} \tau) * (2,1,3)) \underset{i}{\circ} \mathbb{1}_0 = 0$$

for all $1 \leq i \leq 3$. It follows that

$$\tau \underset{1}{\circ} \mathbb{1}_2 - \mathbb{1}_2 \underset{2}{\circ} \tau - (\mathbb{1}_2 \underset{2}{\circ} \tau) * (2,1,3) \in {}^3\Upsilon(3).$$

Since $\mathcal{P}$ is of GK-dimension 3 and ${}^3\Upsilon(3) = 0$, Equation (E3.1.1) holds. Similarly, (E3.1.2), (E3.1.3), (E3.1.4) and (E3.1.5) hold. □

Let $\mathcal{P}$ be a 2-unitary operad with a 2-unit $\mathbb{1}_2$. By convention, we define $\mathbb{1}'_n = \mathbb{1}_n$ for $n = 0, 1, 2$. Recall from [**BYZ**, Section 2] that, for every $n \geq 3$, one can define inductively that

$$\mathbb{1}_n = \mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_{n-1}, \quad \text{and} \quad \mathbb{1}'_n = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}'_{n-1}.$$

By Definition 2.5(4) a 2-unitary operad is called 2a-unitary if $\mathbb{1}_2$ is associative, or equivalently $\mathbb{1}_3 = \mathbb{1}'_3$.

LEMMA 3.2. *Let $\mathcal{P}$ be a 2-unitary operad of GK-dimension 3 with 2-unit $\mathbb{1}_2$. Then $\mathbb{1}_2$ is a 2a-unit. Moreover, if $\mathbb{1}_2$ is a 2a-unit, then so is $\mathbb{1}_2 + \tau$ for any $\tau \in {}^2\Upsilon(2)$.*

PROOF. Suppose that $\mathbb{1}_2$ is a 2-unit of $\mathcal{P}$. By definition, $\mathbb{1}_3 = \mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_2$ and $\mathbb{1}'_3 = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_2$. One can easily check that $(\mathbb{1}_3 - \mathbb{1}'_3) \underset{i}{\circ} \mathbb{1}_0 = 0$ for all $i = 1, 2, 3$. This means that $\mathbb{1}_3 - \mathbb{1}'_3 \in {}^3\Upsilon(3)$. Since $\mathcal{P}$ is of GK-dimension 3, ${}^3\Upsilon(3) = 0$. Thus $\mathbb{1}_3 = \mathbb{1}'_3$ as required.

Clearly, for any $\tau \in {}^2\Upsilon(2)$, we have

$$(\mathbb{1}_2 + \tau) \underset{i}{\circ} \mathbb{1}_0 = \mathbb{1}_2 \underset{i}{\circ} \mathbb{1}_0 = \mathbb{1}_1$$

for $i = 1, 2$. So $\mathbb{1}_2 + \tau$ is a 2-unit. Moreover, by Lemma 3.1 (E3.1.3), we have

$$(\mathbb{1}_2 + \tau) \underset{i}{\circ} (\mathbb{1}_2 + \tau) = \mathbb{1}_2 \underset{i}{\circ} \mathbb{1}_2 + \tau \underset{i}{\circ} \mathbb{1}_2 + \mathbb{1}_2 \underset{i}{\circ} \tau$$

for $i = 1, 2$. Since $\mathbb{1}_2$ is a 2a-unit, $\mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_2$. By direct computation, we have

$$(\mathbb{1}_2 \underset{1}{\circ} \tau + \tau \underset{1}{\circ} \mathbb{1}_2 - \mathbb{1}_2 \underset{2}{\circ} \tau - \tau \underset{2}{\circ} \mathbb{1}_2) \underset{i}{\circ} \mathbb{1}_0 = \tau - \tau = 0$$

for $i = 1, 2, 3$. Therefore, $(\mathbb{1}_2 \underset{1}{\circ} \tau + \tau \underset{1}{\circ} \mathbb{1}_2 - \mathbb{1}_2 \underset{2}{\circ} \tau - \tau \underset{2}{\circ} \mathbb{1}_2) \in {}^3\Upsilon(3)$. Since $\mathcal{P}$ is of GK-dimension 3 and ${}^3\Upsilon(3) = 0$, we know $\mathbb{1}_2 \underset{1}{\circ} \tau + \tau \underset{1}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{2}{\circ} \tau + \tau \underset{2}{\circ} \mathbb{1}_2$. It follows that $\mathbb{1}_2 + \tau$ is a 2a-unit. □

LEMMA 3.3 ([**BYZ**, Lemma 2.7]). *Let $\mathcal{P}$ be a 2a-unitary operad. Then the following hold.*

(1) *For every $n \geq 3$, $\mathbb{1}_n = \mathbb{1}'_n$.*
(2) *For every $n \geq 1$ and $k_1, \cdots, k_n \geq 0$, $\mathbb{1}_n \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_n}) = \mathbb{1}_{k_1+\cdots+k_n}$.*

LEMMA 3.4. *Let $\mathcal{P}$ be a 2-unitary operad of GK-dimension ≤ 3. Suppose that ${}^1\Upsilon(1)$ has a $\Bbbk$-basis $\{\delta_j \mid j \in [d]\}$ and ${}^2\Upsilon(2)$ has a $\Bbbk$-basis $\{\tau_s \mid s \in [m]\}$. Then for every $n \geq 3$, $\mathcal{P}(n)$ has a $\Bbbk$-basis:*

$$\{\mathbb{1}_n\} \cup \{\delta^n_{(i),j} \mid i \in [n], j \in [d]\} \cup \{\tau^n_{(i_1 i_2),s} \mid 1 \leq i_1 < i_2 \leq n, s \in [m]\}, \tag{E3.4.1}$$

where $\delta^n_{(i),j} = (\mathbb{1}_n \underset{1}{\circ} \delta_j) * c_i$, $\tau^n_{(i_1 i_2),s} = (\mathbb{1}_{n-1} \underset{1}{\circ} \tau_s) * c_{i_1 i_2}$, and $c_i = (i, 1, \cdots, i-1, \hat{i}, i+1, \cdots, n)$ and $c_{i_1 i_2} = (i_1, i_2, 1, \cdots, i_1 - 1, \hat{i_1}, i_1 + 1, \cdots, i_2 - 1, \hat{i_2}, i_2 + 1, \cdots, n)$.

PROOF. Since $\mathcal{P}$ is a 2-unitary operad of GK-dimension 3, we have ${}^k\Upsilon(k) = 0$ for all $k \geq 3$. By Theorem 2.8, we can choose a basis of $\mathcal{P}(n)$ as follows

$$\begin{aligned} &\{\mathbb{1}_n\} \cup \{\mathbb{1}_2 \circ (\delta_j, \mathbb{1}_{n-1}) * c_i \mid i \in [n], j \in [d]\} \\ &\qquad \cup \{\mathbb{1}_2 \circ (\tau_s, \mathbb{1}_{n-2}) * c_{i_1 i_2} \mid 1 \leq i_1 < i_2 \leq n, s \in [m]\}. \end{aligned}$$

By Lemmas 3.2 and 3.3

$$\mathbb{1}_n \underset{1}{\circ} \delta_j = \mathbb{1}'_n \underset{1}{\circ} \delta_j = (\mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_{n-1}) \underset{1}{\circ} \delta_j = (\mathbb{1}_2 \underset{1}{\circ} \delta_j) \underset{2}{\circ} \mathbb{1}_{n-1} = \mathbb{1}_2 \circ (\delta_j, \mathbb{1}_{n-1}),$$

$$\mathbb{1}_{n-1} \underset{1}{\circ} \tau_s = \mathbb{1}'_{n-1} \underset{1}{\circ} \tau_s = (\mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_{n-2}) \underset{1}{\circ} \tau_s = (\mathbb{1}_2 \underset{1}{\circ} \tau_s) \underset{3}{\circ} \mathbb{1}_{n-2} = \mathbb{1}_2 \circ (\tau_s, \mathbb{1}_{n-2}),$$

we immediately obtain basis (E3.4.1) of $\mathcal{P}(n)$. □

PROPOSITION 3.5. *Suppose* $\operatorname{char} \Bbbk \neq 2$. *Let $\mathcal{P}$ be a 2-unitary operad of GK-dimension 3 with a 2-unit $\mathbb{1}_2$. Let*

$$\bar{\mathbb{1}}_2 := \frac{1}{2}(\mathbb{1}_2 + \mathbb{1}_2 * (2, 1)).$$

Then $\bar{\mathbb{1}}_2$ is also a 2a-unit. Consequently, $\mathcal{P}$ is $\mathcal{C}om$-augmented.

PROOF. It is easy to check that $\bar{\mathbb{1}}_2$ is a 2-unit. By Lemma 3.2, $\bar{\mathbb{1}}_2$ is a 2a-unit, namely, $(\mathcal{P}, \mathbb{1}_0, \mathbb{1}, \bar{\mathbb{1}}_2)$ is a 2a-unitary operad.

After replacing $\mathbb{1}_2$ by $\bar{\mathbb{1}}_2$ we may assume that $\mathbb{1}_2 * (2, 1) = \mathbb{1}_2$. It follows from induction and Lemma 3.3(1) that $\mathbb{1}_n * \sigma = \mathbb{1}_n$ for all $\sigma \in \mathbb{S}_n$. Therefore there is a canonical morphism from $\mathcal{C}om$ [Example 2.2] to $\mathcal{P}$ sending $\mathbb{1}_n \mapsto \mathbb{1}_n$ for all $n \geq 0$. □

4. Trident algebras

Let R be a untial associative algebra over $\Bbbk$ with a right action of an abelian group G satisfying $(ab).g = (a.g)(b.g)$ and for all $a, b \in R$ and $g \in G$. Such an R is called a $\Bbbk G$*-module algebra* (or G*-module algebra*). Recall that the *skew group algebra* $R\#G$ is the vector space $R \otimes \Bbbk G$ with the multiplication

$$(a\#g)(b\#h) = ((a.h)b)\#(gh)).$$

Furthermore, a right module M over $R\#G$ means that M is a right R-module and a right $\Bbbk G$-module satisfying $(\mu a)g = (\mu g)(a.g)$ for all $\mu \in M, g \in G$ and $a \in R$. The following lemma is easy.

LEMMA 4.1. *Let G be an abelian group. Let A and B be G-module algebras. Suppose M, N are right modules over the skew group algebras $A\#G$ and $B\#G$, respectively. Then $M \otimes N$ is a right $(A \otimes B)\#G$-module with the action given by*

$$(x \otimes y)(a \otimes b\#g) \colon = (x.(a\#g)) \otimes (y.(b\#g))$$

for $x \in M, y \in N, a \in A, b \in B, g \in G$.

Let A be a unital associative algebra. Clearly, the tensor product algebra $A \otimes A$ (also denoted by $A^{\otimes 2}$) admits a natural right $\mathbb{S}_2$-action given by $(a \otimes b)(2,1) \colon = b \otimes a$. So we obtain a skew group algebra $(A \otimes A)\#\mathbb{S}_2$ (also denoted by $A^{\otimes 2}\#\mathbb{S}_2$). Let M be a left A- right $A^{\otimes 2}\#\mathbb{S}_2$-bimodule. Equivalently, M is both a right $\Bbbk\mathbb{S}_2$-module and a left A- right $(A \otimes A)$-bimodule satisfying

$$a(\mu(2,1)) = (a\mu)(2,1), \tag{E4.1.1}$$

$$(\mu.(a \otimes b))(2,1) = (\mu(2,1)).(b \otimes a), \tag{E4.1.2}$$

for all $a, b \in A$, $\mu \in M$.

REMARK 4.2. Observe that if M is both a right $\Bbbk\mathbb{S}_2$-module and a right A-module with the action $M \otimes A \to M, (\mu, a) \mapsto \mu \underset{1}{\cdot} a$, then M admits another right A-module action given by

$$\mu \underset{2}{\cdot} a = ((\mu.(2,1)) \underset{1}{\cdot} a).(2,1),$$

which is called the *congruence action*. Therefore, a right $A^{\otimes 2}\#\mathbb{S}_2$-module action on M is equivalent to a right $\Bbbk\mathbb{S}_2$-action together with a right A-module action satisfying

$$(\mu \underset{1}{\cdot} a) \underset{2}{\cdot} b = (\mu \underset{2}{\cdot} b) \underset{1}{\cdot} a \tag{E4.2.1}$$

for all $\mu \in M, a \in A$.

4.1. Tridents. In this subsection we introduce a new algebraic system.

Let $A = \Bbbk\mathbb{1}_1 \oplus \bar{A}$ be an augmented algebra with augmentation ideal $\bar{A}$. Obviously, the right regular $A^{\otimes 2}$-module $A \otimes A$ with an action of $\mathbb{S}_2$ given by $(a \otimes b)(2,1) = b \otimes a$ admits a right $A^{\otimes 2}\#\mathbb{S}_2$-module structure. Furthermore, its quotient module $(A \otimes A)/(\bar{A} \otimes \bar{A})$ admits an $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodule structure, where the left A-action is given by

$$a \cdot [1_A \otimes 1_A] \colon = [a \otimes 1_A] + [1_A \otimes a],$$

$$a \cdot [b \otimes 1_A] \colon = [(ab) \otimes 1_A], \quad a \cdot [1_A \otimes b] \colon = [1_A \otimes (ab)]$$

for $a, b \in \bar{A}$, where $[x \otimes y]$ denotes the element $x \otimes y + \bar{A} \otimes \bar{A} \in (A \otimes A)/(\bar{A} \otimes \bar{A})$ for $x \otimes y \in A \otimes A$.

In fact, $(A \otimes A)/(\bar{A} \otimes \bar{A})$ is isomorphic to $\Bbbk(1_A \otimes 1_A) \oplus (\bar{A} \otimes \Bbbk 1_A) \oplus (\Bbbk 1_A \otimes \bar{A})$ as a vector space.

Let A be an augmented algebra with the augmented ideal $\bar{A}$, and M be an $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodule. Suppose that E is an extension of M by $(A \otimes A)/(\bar{A} \otimes \bar{A})$ as an $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodule. Then the triple (A, M, E) is called a *trident*.

Let (A, M, E) and (A', M', E') be two tridents. Suppose that $\alpha \colon A \to A'$ is a homomorphism of augmented algebras, and $\beta \colon M \to M'$ is a homomorphism of $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodules with the $(A, A^{\otimes 2}\#\mathbb{S}_2)$-action on M' given by the algebra homomorphisms $\alpha \colon A \to A'$ and $\alpha \otimes \alpha \colon A \otimes A \to A' \otimes A'$. Clearly, $[\alpha \otimes \alpha] \colon (A \otimes$

$A)/(\bar{A}\otimes\bar{A}) \to (A'\otimes A')/(\bar{A}'\otimes\bar{A}')$ is a homomorphism of $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodules. Then one can obtain a homomorphism of extensions of $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & (A\otimes A)/(\bar{A}\otimes\bar{A}) & \longrightarrow & 0 \\ & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle[\alpha\otimes\alpha]} & & \\ 0 & \longrightarrow & M' & \longrightarrow & E' & \longrightarrow & (A'\otimes A')/(\bar{A}'\otimes\bar{A}') & \longrightarrow & 0 \end{array}$$

which is also called a homomorphism of tridents. Consequently, we obtain a category $\mathcal{T}$, called the *trident category*. The following lemma follows from Lemma 4.1.

LEMMA 4.3. *Let A, B be associative algebras over $\Bbbk$. Suppose that M and N are $(A, A^{\otimes 2}\#\mathbb{S}_2)$- and $(B, B^{\otimes 2}\#\mathbb{S}_2)$-bimodules, respectively. Then $M\otimes N$ is an $(A\otimes B, (A\otimes B)^{\otimes 2}\#\mathbb{S}_2)$-bimodule.*

4.2. Trident systems. There is another way of introducing a trident. Denote by mod-$\mathbb{S}_2$ the category of finite dimensional right $\Bbbk\mathbb{S}_2$-modules. It is well known that $V\otimes W\in$ mod-$\mathbb{S}_2$ with the diagonal action

$$(v\otimes w)*(2,1) = (v*(2,1))\otimes(w*(2,1))$$

for $V, W \in$ mod-$\mathbb{S}_2$. Let B be an associative algebra in the category mod-$\mathbb{S}_2$. Such an algebra is also called a $\Bbbk\mathbb{S}_2$-module algebra (in Hopf algebra language). Sometimes it is called an algebra with involution (but $\Bbbk$ may not be the complex field). A right (*resp.* left) module V over a $\Bbbk\mathbb{S}_2$-module algebra B means $V\in$ mod-$\mathbb{S}_2$ and the action $V\otimes B\to V$ (*resp.* $B\otimes V\to V$) is a homomorphism of $\Bbbk\mathbb{S}_2$-modules.

DEFINITION 4.4. A pair (A, M) with morphisms f, g, or equivalently, a quadruple (A, M, f, g), is called a *trident system* if

(10) $A = \Bbbk\mathbb{1}_1\oplus\bar{A}$ is an augmented algebra with the augmentation ideal $\bar{A}$,
(20) M is an $(A, A\otimes A)$-bimodule in mod-$\mathbb{S}_2$,
(30) $f\colon \bar{A}\to M$ is a $\Bbbk$-linear map in mod-$\mathbb{S}_2$ where the $\mathbb{S}_2$-action on $\bar{A}$ is trivial,
(40) $g\colon \bar{A}\otimes\bar{A}\to M$ is a homomorphism of right $A\otimes A$-modules in mod-$\mathbb{S}_2$,

such that the following identities hold

$$f(ab) = af(b) + f(a)\cdot(b\otimes 1_A) + f(a)\cdot(1_A\otimes b) + g(a,b) + g(b,a), \tag{E4.4.1}$$

$$f(a)\cdot(b,c) = ag(b,c) - g(ab,c) - g(b,ac), \tag{E4.4.2}$$

for all $a, b, c\in\bar{A}$.

We define morphisms between trident systems as follows. Let (A, M, f, g) and (A', M', f', g') be two trident systems. A morphism $(\alpha,\beta)\colon (A, M, f, g)\to (A', M', f', g')$ is given by an algebra homomorphism $\alpha\colon A\to A'$ and a trident A-module homomorphism $\beta\colon M\to M'$ such that the following diagrams commute

$$\begin{array}{ccc} \bar{A} & \xrightarrow{f} & M \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ \bar{A}' & \xrightarrow{f'} & M' \end{array} \qquad\text{and}\qquad \begin{array}{ccc} \bar{A}\otimes\bar{A} & \xrightarrow{g} & M \\ {\scriptstyle\alpha\otimes\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ \bar{A}'\otimes\bar{A}' & \xrightarrow{g'} & M' \end{array}$$

where the $(A, A\otimes A)$-bimodule structure on M' is determined by $(A', A'\otimes A')$-bimodule action and the algebra homomorphisms $\alpha\colon A\to A'$ and $\alpha\otimes\alpha\colon A\otimes A\to A'\otimes A'$.

One can define a category $\mathcal{C}$ consisting of all trident systems and morphisms defined above.

PROPOSITION 4.5. *Retain the above notation. The trident category is isomorphic to the category of trident systems.*

PROOF. Let (A, M, f, g) be a trident system. Recall that $A^{\otimes 2}$ is a subring of $A^{\otimes 2}\#\mathbb{S}_2$. Then f and g satisfy

(1) $f(a) = f(a) * (2,1)$,
(2) $g(a,b) = g(b,a) * (2,1)$,
(3) $f(ab) = a \cdot f(b) + f(a) \cdot (b \otimes 1_A\#(1)) + f(a) \cdot (1_A \otimes b\#(1)) + g(a,b) + g(b,a)$,
(4) $f(a) \cdot (b \otimes c\#(1)) = a \cdot g(b,c) - g(ab,c) - g(b,ac)$,

for all $a, b, c \in \bar{A}$. Using these equations, one can define an extension E of M by $(A \otimes A)/(\bar{A} \otimes \bar{A})$. To be precise, $E = M \oplus (A \otimes A)/(\bar{A} \otimes \bar{A})$ as a right $\mathbb{S}_2$-module with the $(A, A \otimes A)$-bimodule action given by

$$\begin{aligned} &a \cdot (x, \lambda[1_A \otimes 1_A] + [b \otimes 1_A] + [1_A \otimes c]) \\ &\quad = (ax + \lambda f(a) + f(a) \cdot (b \otimes 1_A) + f(a) \cdot (1_A \otimes c) + g(b,a) + g(a,c), \\ &\qquad\qquad \lambda[a \otimes 1_A] + \lambda[1_A \otimes a] + [ab \otimes 1_A] + [1_A \otimes ac]), \end{aligned}$$

$$\begin{aligned} &(x, \lambda[1_A \otimes 1_A] + [b \otimes 1_A] + [1_A \otimes c]) \cdot (a \otimes 1_A) \\ &\qquad = (x \cdot (a \otimes 1_A) + g(a,c), \lambda[a \otimes 1_A] + [ba \otimes 1_A]), \end{aligned}$$

$$\begin{aligned} &(x, \lambda[1_A \otimes 1_A] + [b \otimes 1_A] + [1_A \otimes c]) \cdot (1_A \otimes a) \\ &\qquad = (x \cdot (1_A \otimes a) + g(b,a), \lambda[1_A \otimes a] + [1 \otimes ca]), \end{aligned}$$

$$\begin{aligned} &(x, \lambda[1_A \otimes 1_A] + [b \otimes 1_A] + [1_A \otimes c]) \cdot (a \otimes a') \\ &\qquad = (x \cdot (a \otimes a') + \lambda g(a,a') + g(ba,a') + g(a,ca'), 0) \end{aligned}$$

for all $\lambda \in \mathbb{k}$, $a, b, c \in \bar{A}$ and $x \in M$. It is easy to verify that (A, M, E) is a trident.

Conversely, given a trident (A, M, E), we construct a trident system as follows. Suppose that

$$0 \to M \to E \xrightarrow{\pi} (A \otimes A)/(\bar{A} \otimes \bar{A}) \to 0$$

is the corresponding short exact sequence of $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodules. Without loss of generality, we assume M is a sub-bimodule of E. Fix an element $\mathbb{1}_2 \in E$ with $\pi(\mathbb{1}_2) = [1_A \otimes 1_A] \in (A \otimes A)/(\bar{A} \otimes \bar{A})$. For all $a, b \in \bar{A}$, we define

$$\begin{aligned} f(a) &:= a \cdot \mathbb{1}_2 - \mathbb{1}_2 \cdot (a \otimes 1_A\#(1)) - \mathbb{1}_2 \cdot (1_A \otimes a\#(1)), \\ g(a,b) &:= \mathbb{1}_2 \cdot (a \otimes b\#(1)) \end{aligned}$$

in E. Clearly, in $(A \otimes A)/(\bar{A} \otimes \bar{A})$, we have

$$\begin{aligned} \pi(f(a)) &= a \cdot [1_A \otimes 1_A] - [a \otimes 1_A] - [1_A \otimes a] = 0, \\ \pi(g(a,b)) &= [1_A \otimes 1_A] \cdot (a \otimes b) = 0 \end{aligned}$$

Therefore, we obtain two $\mathbb{k}$-linear maps

$$f : \bar{A} \to M, \quad \text{and} \quad g : \bar{A} \otimes \bar{A} \to M.$$

It can be directly checked that (A, M, f, g) is a trident system.

Since both constructions above are canonical, these defines two functors that are inverse to each other. $\square$

DEFINITION 4.6. A *trident algebra* means either a trident (A, M, E) or a trident system (A, M, f, g).

4.3. Examples. We give some examples of trident algebras.

EXAMPLE 4.7. Let A be an augmented algebra and let M be an $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodule. Consider the trivial extension of M by $(A \otimes A)/(\bar{A} \otimes \bar{A})$. Equivalently, $f : \bar{A} \to M$ and $g : \bar{A}^{\otimes 2} \to M$ are zero maps in the corresponding trident system. In this case, $(A, M, 0, 0)$ is called a *trivial* trident algebra.

In particular, in case $A = \Bbbk$, any trident algebra is isomorphic to $(\Bbbk, M, 0, 0)$ for some right $\mathbb{S}_2$-module M and hence trivial. Also see Corollary 6.1.

Next we show some examples of nontrivial and natural trident algebras.

EXAMPLE 4.8. Let A be an augmented algebra and let $E = A \otimes A \otimes A$ be the $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodule with the actions given by

$$a \cdot (x \otimes y \otimes z) \cdot (b \otimes c\#(2, 1))\colon = ax \otimes zb \otimes yc$$

for all $a \in A, x \otimes y \otimes z \in E, b \otimes c\#(2, 1) \in A^{\otimes 2}\#\mathbb{S}_2$. Consider the homomorphism $\pi\colon E \to (A \otimes A)/(\bar{A} \otimes \bar{A})$ of $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodules induced by

$$\pi(1_A \otimes 1_A \otimes 1_A) = [1_A \otimes 1_A].$$

Clearly, π is an epimorphism. Take $M = \operatorname{Ker}\pi$, and we obtain a trident algebra (A, M, E). Equivalently, we have a trident system $(A, \operatorname{Ker}\pi, f, g)$ with

$$f(a) = a \otimes 1_A \otimes 1_A - 1_A \otimes a \otimes 1_A - 1_A \otimes 1_A \otimes a,$$

$$g(b \otimes c) = 1_A \otimes b \otimes c,$$

for all $a \in \bar{A}, b \otimes c \in \bar{A} \otimes \bar{A}$.

We construct another related trident algebra. Let J be the sub-bimodule of E generated by

$$\{a \otimes 1_A \otimes 1_A - 1_A \otimes a \otimes 1_A - 1_A \otimes 1_A \otimes a \mid a \in \bar{A}\}.$$

Clearly, $J \subset \operatorname{Ker}\pi$. Set $E' = E/J$. Then we have an epimorphism

$$\pi'\colon E' \to (A \otimes A)/(\bar{A} \otimes \bar{A})$$

of $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodules, and hence a trident algebra $(A, \operatorname{Ker}\pi', E')$. This trident algebra corresponds to the trident system $(A, \operatorname{Ker}\pi', f', g')$ with

$$f' = 0, \quad g'(b \otimes c) = \overline{1_A \otimes b \otimes c}$$

for all $b \otimes c \in \bar{A} \otimes \bar{A}$.

EXAMPLE 4.9. This is the trident algebra corresponding to $\mathcal{D}_A \underset{\mathrm{H}}{\otimes} \mathcal{D}_B$, where $\mathcal{D}_A$ and $\mathcal{D}_B$ are the 2-unitary operad defined in [**BYZ**, Example 2.4].

Let A and B be augmented algebras. Clearly, $A \otimes B$ is also an augmented algebra with $\overline{A \otimes B} := \bar{A} \otimes \Bbbk 1_B + \Bbbk 1_A \otimes \bar{B} + \bar{A} \otimes \bar{B}$. From Subsection 4.1, we know that $(A \otimes A)/(\bar{A} \otimes \bar{A})$ and $(B \otimes B)/(\bar{B} \otimes \bar{B})$ are $(A, A^{\otimes 2}\#\mathbb{S}_2)$-bimodule and $(B, B^{\otimes 2}\#\mathbb{S}_2)$-bimodule, respectively. By Lemma 4.1, we obtain an $(A \otimes B, (A \otimes B)^{\otimes 2}\#\mathbb{S}_2)$-bimodule

$$E = \{(A \otimes A)/(\bar{A} \otimes \bar{A})\} \otimes \{(B \otimes B)/(\bar{B} \otimes \bar{B})\}.$$

Observe that,

$$E = (\Bbbk[1_A \otimes 1_A] \oplus (\bar{A} \otimes \Bbbk 1_A) \oplus (\Bbbk 1_A \otimes \bar{A}))\otimes \\ (\Bbbk[1_B \otimes 1_B] \oplus (\bar{B} \otimes \Bbbk 1_B) \oplus (\Bbbk 1_B \otimes \bar{B}))$$

and

$$M := \{(\bar{A} \otimes \Bbbk 1_A) \otimes (\Bbbk 1_B \otimes \bar{B})\} \oplus \{(\Bbbk 1_A \otimes \bar{A}) \otimes (\bar{B} \otimes \Bbbk 1_B)\}$$

is a sub-bimodule of E. By an easy computation, we have

$$\begin{aligned} E/M \cong & \Bbbk[1_{A\otimes B} \otimes 1_{A\otimes B}] \oplus [(\Bbbk 1_A \otimes \bar{B}) \otimes \Bbbk 1_{A\otimes B}] \oplus [(\Bbbk 1_{A\otimes B} \otimes (\Bbbk 1_A \otimes \bar{B}))] \\ & \oplus [(\bar{A} \otimes \Bbbk 1_B) \otimes \Bbbk 1_{A\otimes B}] \oplus [(\bar{A} \otimes \bar{B}) \otimes \Bbbk 1_{A\otimes B}] \\ & \oplus [\Bbbk 1_{A\otimes B} \otimes (\bar{A} \otimes \Bbbk 1_B)] \oplus [\Bbbk 1_{A\otimes B} \otimes (\bar{A} \otimes \bar{B})] \\ = & \Bbbk[1_{A\otimes B} \otimes 1_{A\otimes B}] \oplus (\overline{A \otimes B} \otimes \Bbbk 1_{A\otimes B}) \oplus (\Bbbk 1_{A\otimes B} \otimes \overline{A \otimes B}) \\ = & [(A \otimes B) \otimes (A \otimes B)]/[(\overline{A \otimes B}) \otimes (\overline{A \otimes B})] \end{aligned}$$

Therefore, we obtain a trident $(A \otimes B, M, E)$ which is denoted by $A \odot B$. Using the language of trident system, we have

(1) $f\colon \overline{A \otimes B} \to M$ is determined by

$$\begin{aligned} f(a \otimes 1_B) &= 0, \\ f(1_A \otimes b) &= 0, \\ f(a \otimes b) &= (a \otimes 1_A) \otimes (1_B \otimes b) + (1_A \otimes a) \otimes (b \otimes 1_B) \end{aligned}$$

for all $a \in \bar{A}$ and $b \in \bar{B}$.

(2) $g\colon \overline{A \otimes B} \otimes \overline{A \otimes B} \to M$ is determined by

$$\begin{aligned} g((a \otimes 1_B) \otimes (a' \otimes 1_B)) &= 0, \\ g((a \otimes 1_B) \otimes (1_A \otimes b')) &= (a \otimes 1_A) \otimes (1_B \otimes b'), \\ g((a \otimes 1_B) \otimes (a' \otimes b')) &= 0, \\ g((1_A \otimes b) \otimes (a' \otimes 1_B)) &= (1_A \otimes a') \otimes (b \otimes 1_B), \\ g((1_A \otimes b) \otimes (1_A \otimes b')) &= 0, \\ g((1_A \otimes b) \otimes (a' \otimes b')) &= 0, \\ g((a \otimes b) \otimes (a' \otimes 1_B)) &= 0, \\ g((a \otimes b) \otimes (1_A \otimes b')) &= 0, \\ g((a \otimes b) \otimes (a' \otimes b')) &= 0 \end{aligned}$$

for all $a, a' \in \bar{A}$ and $b, b' \in \bar{B}$.

5. Classification of 2-unitary operads of GK-dimension 3

5.1. An operad constructed from a trident algebra. In this subsection, we construct a 2-unitary operad $\mathcal{P}$ by (A, M, E), with $\mathcal{P}(0) = \Bbbk \mathbb{1}_0$, $\mathcal{P}(1) = A$ and $\mathcal{P}(2) = E$, where the composition $\mathcal{P}(1) \underset{1}{\circ} \mathcal{P}(1) \to \mathcal{P}(1)$ is given by the multiplication of A, the compositions $\mathcal{P}(1) \underset{1}{\circ} \mathcal{P}(2) \to \mathcal{P}(2)$, $\mathcal{P}(2) \circ (\mathcal{P}(1), \mathcal{P}(1)) \to \mathcal{P}(2)$ are given by the corresponding actions of A on E.

Precisely, given a trident system (A, M, f, g) (where $A = \Bbbk \oplus \bar{A}$), consider the operad generated by the $\Bbbk\mathbb{S}$-module $(\Bbbk\mathbb{1}_0, A, \Bbbk\mathbb{1}_2 \oplus M, 0, 0, \cdots)$ and subject to the following relations

$$\begin{gathered}
a \circ \mathbb{1}_0 = 0, \text{ for all } a \in \bar{A}, \\
\mu \underset{i}{\circ} \mathbb{1}_0 = 0, \text{ for all } \mu \in M, \\
\mathbb{1}_2 \underset{i}{\circ} \mathbb{1}_0 = \mathbb{1}_1, \text{ for } i = 1, 2, \\
a \circ b = ab, \text{ for all } a, b \in \bar{A}, \\
a \circ \mu = a \cdot \mu, \text{ for all } a \in \bar{A}, \mu \in M, \\
a \circ \mathbb{1}_2 = \mathbb{1}_2 \underset{1}{\circ} a + \mathbb{1}_2 \underset{2}{\circ} a + f(a), \text{ for all } a \in \bar{A}, \\
\mathbb{1}_2 \circ (a, b) = g(a, b), \text{ for all } a, b \in \bar{A}, \\
\mu \circ (a, b) = \mu \cdot (a \otimes b), \text{ for all } \mu \in M, a, b \in \bar{A}, \\
\mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_2, \\
\mu \underset{1}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{2}{\circ} \mu + (\mathbb{1}_2 \underset{2}{\circ} \mu) * (2, 1, 3), \text{ for all } \mu \in M, \\
\mu \underset{2}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{1}{\circ} \mu + (\mathbb{1}_2 \underset{1}{\circ} \mu) * (1, 3, 2), \text{ for all } \mu \in M, \\
\mathbb{1}_2 \circ (\mu, a) = 0, \text{ for all } \mu \in M, a \in \bar{A}, \\
\mu \underset{i}{\circ} \mu' = 0, \text{ for all } \mu, \mu' \in M, i = 1, 2,
\end{gathered}$$

where $\mathbb{1}_2 * (2, 1) = \mathbb{1}_2$, and $ab, a \cdot \mu, \mu \cdot (a \otimes b)$ are given by the multiplication of A, the left module action of A on M, and the right module action of $A \otimes A$ on M, respectively.

Next we give an explicit description of the above operad. Let $\mathcal{P} = F(A, M, f, g)$ be the following $\Bbbk\mathbb{S}$-module with partial compositions.

(C1) The vector space $\mathcal{P}(n)$:

(C1.1) $\mathcal{P}(0) = \Bbbk\mathbb{1}_0$.

(C1.2) $\mathcal{P}(1) = A = \Bbbk\mathbb{1}_1 \oplus \bar{A}$.

(C1.3) for each $n \geq 2$,

$$\mathcal{P}(n) = \Bbbk\mathbb{1}_n \oplus \bigoplus_{k=1}^{n} \bar{A}_k^{(n)} \oplus \bigoplus_{1 \leq i < j \leq n} M_{ij}^{(n)},$$

where $\Bbbk\mathbb{1}_n$ is a 1-dimensional vector space with the basis $\mathbb{1}_n$, $\bar{A}_k^{(n)}$ is a vector space isomorphic to $\bar{A}$ for $1 \leq k \leq n$, $n \geq 2$, and $M_{ij}^{(n)}$ is a vector space isomorphic to M for $1 \leq i < j \leq n$, $n \geq 2$. By convention $M_{12}^{(2)} = M$. So $\mathcal{P}(2) = \Bbbk\mathbb{1}_2 \oplus \bar{A}_1^{(1)} \oplus \bar{A}_2^{(1)} \oplus M$. In order to write elements in $\bar{A}_k^{(n)}$ and $M_{ij}^{(n)}$, we fix two families of $\Bbbk$-linear isomorphisms

$$\varphi_k^n \colon \bar{A} \to \bar{A}_k^{(n)} \quad \text{and} \quad \psi_{ij}^n \colon M \to M_{ij}^{(n)},$$

for $1 \leq k \leq n, 1 \leq i < j \leq n$ and $n \geq 2$. In fact, we will see that $\bar{A}_k^{(n)} = \{\mathbb{1}_n \underset{k}{\circ} a \mid a \in \bar{A}\}$ and $M_{ij}^{(n)} = \{(\mathbb{1}_{n-1} \underset{1}{\circ} \mu) * c_{ij} \mid \mu \in M\}$, where $c_{ij} = (i, j, 1, \cdots, i-1, \hat{i}, i+1, \cdots, j-1, \hat{j}, j+1, \cdots, n)$.

(C2) The right action of $\Bbbk\mathbb{S}_n$ on $\mathcal{P}(n)$: for each $\sigma \in \mathbb{S}_n$,

(C2.1) $\mathbb{1}_n * \sigma = \mathbb{1}_n$, (C2.2) $\varphi_i^{(n)}(a) * \sigma = \varphi_{\sigma^{-1}(i)}^{(n)}(a)$,

(C2.3) $\psi_{ij}^{(n)}(\mu) * \sigma = \begin{cases} \psi_{\sigma^{-1}(i),\sigma^{-1}(j)}^{(n)}(\mu), & \sigma^{-1}(i) < \sigma^{-1}(j), \\ \psi_{\sigma^{-1}(j),\sigma^{-1}(i)}^{(n)}(\mu * (2,1)), & \sigma^{-1}(i) > \sigma^{-1}(j). \end{cases}$

(C3) The partial composition $\mathcal{P}(m) \underset{s}{\circ} \mathcal{P}(n) \to \mathcal{P}(m+n-1)$:

(C3.1) $\mathbb{1}_m \underset{s}{\circ} \mathbb{1}_n = \mathbb{1}_{m+n-1}$.

(C3.2) $\mathbb{1}_m \underset{s}{\circ} \varphi_i^{(n)}(a) = \varphi_{s+i-1}^{(m+n-1)}(a)$.

(C3.3) $\mathbb{1}_m \underset{s}{\circ} \psi_{i_1,i_2}^{(n)}(\mu) = \psi_{s+i_1-1,s+i_2-1}^{(m+n-1)}(\mu)$.

(C3.4)

$$\varphi_i^{(m)}(a) \underset{s}{\circ} \mathbb{1}_n = \begin{cases} \varphi_i^{(m+n-1)}(a), & i < s, \\ \sum_{k=i}^{i+n-1} \varphi_k^{(m+n-1)}(a) + \sum_{i \le k_1 < k_2 \le i+n-1} \psi_{k_1 k_2}^{(m+n-1)}(f(a)), & i = s, \\ \varphi_{i+n-1}^{(m+n-1)}(a), & i > s. \end{cases}$$

(C3.5)

$$\varphi_i^{(m)}(a) \underset{s}{\circ} \varphi_j^{(n)}(b) = \begin{cases} \psi_{i,s+j-1}^{(m+n-1)}(g(a,b)), & i < s, \\ \sum_{k=i}^{i+j-2} \psi_{k,i+j-1}^{(m+n-1)}(g(a,b) + f(a) \underset{2}{\cdot} b) + \varphi_{i+j-1}^{(m+n-1)}(ab) & \\ \quad + \sum_{k=i+j}^{i+n-1} \psi_{i+j-1,k}^{(m+n-1)}(g(b,a) + f(a) \underset{1}{\cdot} b), & i = s, \\ \psi_{s+j-1,i+n-1}^{(m+n-1)}(g(b,a)), & i > s. \end{cases}$$

(C3.6) $\varphi_i^{(m)}(a) \underset{s}{\circ} \psi_{j_1 j_2}^{(n)}(\mu) = \begin{cases} 0, & i \neq s, \\ \psi_{i+j_1-1,i+j_2-1}^{(m+n-1)}(a\mu), & i = s. \end{cases}$

(C3.7) $\psi_{i_1 i_2}^{(m)}(\mu) \underset{s}{\circ} \mathbb{1}_n = \begin{cases} \psi_{i_1+n-1,i_2+n-1}^{(m+n-1)}(\mu), & 1 \le s < i_1, \\ \sum_{k=i_1}^{i_1+n-1} \psi_{k,i_2+n-1}^{(m+n-1)}(\mu), & s = i_1, \\ \psi_{i_1,i_2+n-1}^{(m+n-1)}(\mu), & i_1 < s < i_2, \\ \sum_{k=i_2}^{i_2+n-1} \psi_{i_1,k}^{(m+n-1)}(\mu), & s = i_2, \\ \psi_{i_1 i_2}^{(m+n-1)}(\mu), & i_2 < s \le m. \end{cases}$

(C3.8) $\psi_{i_1 i_2}^{(m)}(\mu) \underset{s}{\circ} \varphi_j^{(n)}(b) = \begin{cases} 0, & s \neq i_1, i_2, \\ \psi_{i_1+j-1,i_2+n-1}^{(m+n-1)}(\mu \underset{1}{\cdot} b), & s = i_1, \\ \psi_{i_1,i_2+j-1}^{(m+n-1)}(\mu \underset{2}{\cdot} b), & s = i_2. \end{cases}$

(C3.9) $\psi_{i_1 i_2}^{(m)}(\mu) \underset{s}{\circ} \psi_{j_1 j_2}^{(n)}(\nu) = 0$.

Theorem 5.1. *Retain the above notation. Let (A, M, f, g) be a trident algebra. Suppose $M \neq 0$. Then $F(A, M, f, g)$ is a 2-unitary $\mathcal{C}om$-augmented operad of GK-dimension ≤ 3. Further, $\operatorname{GKdim} F(A, M, f, g) \le 2$ if and only if $M = 0$.*

Proof. We firstly check that $\mathcal{P} := F(A, M, f, g)$ is an operad. Set $\mathbb{A} = \{\mathbb{1}_n\}_{n=0}^{\infty}$, $\Phi = \{\varphi_i^{(n)}(a) \mid a \in \bar{A}\}_{i,n}$, and $\Psi = \{\psi_{ij}^{(n)}(y) \mid y \in M\}_{i,j,n}$. It suffices to show that (OP1), (OP2) and (OP3) hold for these elements.

Verification of (OP1): By (C3.1), (C3.2) and (C3.3), $\mathbb{1} \underset{1}{\circ} \theta = \theta$ for all $\theta \in \mathcal{P}$. By (C3.1), (C3.4) and (C3.7), we have $\theta \underset{i}{\circ} \mathbb{1} = \theta$ for all $\theta \in \mathcal{P}$ and $1 \le i \le \mathrm{Ar}(\theta)$. Therefore (OP1) holds.

Verification of (OP2): We make a discussion on λ, μ, ν. Clearly we have the following types, Ψ^3, $\Phi\Psi^2$, $\mathbb{A}\Psi^2$, $\Phi^2\Psi$, $\mathbb{A}\Phi\Psi$, $\mathbb{A}^2\Psi$, Φ^3, $\mathbb{A}\Phi^2$, $\mathbb{A}^2\Phi$, $\mathbb{A}^3$. Here for instance, by the type $\Phi\Psi^2$ we mean that one of λ, μ, ν is taken from Φ and the other two are from Ψ. Other types are similar.

For each type there are several subcases, for instance, the case of type $\Phi\Psi^2$ has 3 subcases, say $\lambda \in \Phi, \mu, \nu \in \Psi$, or $\mu \in \Phi, \lambda, \nu \in \Psi$, or $\nu \in \Phi, \lambda, \mu \in \Psi$. Altogether we have 27 subcases.

By (C3.7), (C3.8) and (C3.9), if at least two of λ, μ, ν are from Ψ, say in the types Ψ^3, $\Phi\Psi^2$, $\mathbb{A}\Psi^2$, then both sides of (E2.1.1) and (E2.1.2) are obviously 0 and hence (OP2) holds automatically. Thus there are 20 subcases left. The proof is routine but tedious and we only check a special case, namely, (E2.1.1) holds for $\lambda \in \Psi$ and $\mu, \nu \in \Phi$, and other cases can be checked similarly.

Assume that $\lambda = \psi^{(l)}_{k_1k_2}(y)$, $\mu = \varphi^{(m)}_s(a)$, and $\nu = \varphi^{(n)}_t(b)$. Then

$$\text{LHS of (E2.1.1)} = (\psi^{(l)}_{k_1k_2}(y) \underset{i}{\circ} \varphi^{(m)}_s(a)) \underset{i-1+j}{\circ} \varphi^{(n)}_t(b)$$

$$\underset{\text{by (C3.8)}}{=} \begin{cases} 0 & i \neq k_1, k_2, \\ \psi^{(l+m-1)}_{k_1+s-1,k_2+m-1}(y \underset{1}{\cdot} a) & i = k_1, \\ \psi^{(l+m-1)}_{k_1,k_2+s-1}(y \underset{2}{\cdot} a) & i = k_2, \end{cases} \qquad \underset{i-1+j}{\circ} \varphi^{(n)}_t(b)$$

$$\underset{\text{by (C3.8)}}{=} \begin{cases} 0 & i \neq k_1, k_2, i+j-1 \neq k_1+s-1, k_2+m-1, k_1, k_2+s-1, \\ \psi^{(l+m+n-2)}_{k_1+s+t-2,k_2+m+n-2}((y \underset{1}{\cdot} a) \underset{1}{\cdot} b) & i = k_1, i+j-1 = k_1+s-1, \\ \psi^{(l+m+n-2)}_{k_1+s-1,k_2+m+t-2}((y \underset{1}{\cdot} a) \underset{2}{\cdot} b) & i = k_1, i+j-1 = k_2+m-1, \ (\textit{impossible}) \\ \psi^{(l+m+n-2)}_{k_1+t-1,k_2+s+n-2}((y \underset{2}{\cdot} a) \underset{1}{\cdot} b) & i = k_2, i+j-1 = k_1, \ (\textit{impossible}) \\ \psi^{(l+m+n-2)}_{k_1,k_2+s+t-2}((y \underset{2}{\cdot} a) \underset{2}{\cdot} b) & i = k_2, i+j-1 = k_2+s-1, \end{cases}$$

$$= \begin{cases} \psi^{(l+m+n-2)}_{k_1+s+t-2,k_2+m+n-2}((y \underset{1}{\cdot} a) \underset{1}{\cdot} b) & i = k_1, j = s, \\ \psi^{(l+m+n-2)}_{k_1,k_2+s+t-2}((y \underset{2}{\cdot} a) \underset{2}{\cdot} b) & i = k_2, j = s, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\text{RHS of (E2.1.1)} = \psi^{(l)}_{k_1k_2}(y) \underset{i}{\circ} (\varphi^{(m)}_s(a) \underset{j}{\circ} \varphi^{(n)}_t(b))$$

$$\underset{\text{by (C3.5)}}{=} \psi^{(l)}_{k_1k_2}(y) \underset{i}{\circ} \begin{cases} \psi^{(m+n-1)}_{s,j+t-1}(g(a,b)), & s < j, \\ \sum\limits_{k=s}^{s+t-2} \psi^{(m+n-1)}_{k,s+t-1}(g(a,b) + f(a) \underset{2}{\cdot} b) + \varphi^{(m+n-1)}_{s+t-1}(ab) & \\ \quad + \sum\limits_{k=s+t}^{s+n-1} \psi^{(m+n-1)}_{s+t-1,k}(g(b,a) + f(a) \underset{1}{\cdot} b), & s = j, \\ \psi^{(m+n-1)}_{j+t-1,s+n-1}(g(b,a)), & s > j. \end{cases}$$

$$\underset{\text{by (C3.9)}}{=} \begin{cases} 0 & s < j, \\ \psi^{(l)}_{k_1k_2}(y) \underset{i}{\circ} \varphi^{(m+n-1)}_{s+t-1}(ab) & s = j, \\ 0 & s > j, \end{cases}$$

$$\underset{\text{by (C3.6)}}{=} \begin{cases} 0 & s < j, \\ 0 & s = j, i \neq k_1, k_2, \\ \psi^{(l+m+n-2)}_{k_1+s+t-2,k_2+m+n-2}(y \underset{1}{\cdot} ab) & s = j, i = k_1 \\ \psi^{(l+m+n-2)}_{k_1,k_2+s+t-2}(y \underset{2}{\cdot} ab) & s = j, i = k_2 \\ 0 & s > j, \end{cases}$$

$$= \begin{cases} \psi^{(l+m+n-2)}_{k_1+s+t-2,k_2+m+n-2}((y \underset{1}{\cdot} a) \underset{1}{\cdot} b) & i = k_1, j = s, \\ \psi^{(l+m+n-2)}_{k_1,k_2+s+t-2}((y \underset{2}{\cdot} a) \underset{2}{\cdot} b) & i = k_2, j = s, \\ 0 & \text{otherwise}, \end{cases}$$

which implies that (E2.1.1) holds.

Verification of (OP3): Since each of μ and ν has 3 choices, say, belongs to $\mathbb{A}$, Φ or Ψ, we have 9 cases. (OP3) for the case $\mu, \nu \in \mathbb{A}$ is trivial and we have 8 cases left. We only check a special case, namely, (E2.1.3) for $\mu \in \Phi$ and $\nu \in \mathbb{A}$, and $\sigma \in \mathbb{S}_n$, and other cases can be checked similarly. Recall that $\sigma' = 1_m \underset{i}{\circ} \sigma$. Write

$$\sigma = (k_1,\ k_2,\ \cdots,\ k_n) \tag{E5.1.1}$$

where by convention $k_w = \sigma^{-1}(w)$ for all w. Then, by definition,
(E5.1.2)

$$\sigma' = (1, \cdots, i-1, k_1 + i - 1, k_2 + i - 1, \cdots, k_n + i - 1, i + n, \cdots, n + m - 1).$$

By (E5.1.2), $(\sigma')^{-1}(s) = s$ if $s < i$ and $s \geq i+n$ and $(\sigma')^{-1}(k+i-1) = \sigma^{-1}(k)+i-1$ for $1 \leq k \leq n$. We refer to [**BYZ**, Section 8] for more details concerning σ'.

Write $\mu = \varphi_k^{(m)}(a)$ and $\nu = \mathbb{1}_n$. Then

$$\begin{aligned}
\text{LHS of (E2.1.3)} &= \varphi_k^{(m)}(a) \underset{i}{\circ} (\mathbb{1}_n * \sigma) \underset{\text{by (C3.3)}}{=} \varphi_k^{(m)}(a) \underset{i}{\circ} \mathbb{1}_n \\
&\underset{\text{by (C3.4)}}{=} \begin{cases} \varphi_k^{(m+n-1)}(a), & k < i, \\ \sum\limits_{w=k}^{k+n-1} \varphi_k^{(m+n-1)}(a) + \sum\limits_{k \leq k_1 < k_2 \leq k+n-1} \psi_{k_1 k_2}^{(m+n-1)}(f(a)), & k = i, \\ \varphi_{k+n-1}^{(m+n-1)}(a), & k > i. \end{cases}
\end{aligned}$$

and in the following computation we use the fact that $f(a)*(2,1) = f(a)$ [Definition 4.4(3)] and notation $(k_1', k_2') = ((\sigma')^{-1}(k_1), (\sigma')^{-1}(k_2))$ or $((\sigma')^{-1}(k_2), (\sigma')^{-1}(k_1))$,

$$\begin{aligned}
\text{RHS of (E2.1.3)} &= (\varphi_k^{(m)}(a) \underset{i}{\circ} \mathbb{1}_n) * \sigma' \\
&\underset{\text{by (C3.4)}}{=} \begin{cases} \varphi_k^{(m+n-1)}(a), & k < i, \\ \sum\limits_{w=k}^{k+n-1} \varphi_k^{(m+n-1)}(a) + \sum\limits_{k \leq k_1 < k_2 \leq k+n-1} \psi_{k_1 k_2}^{(m+n-1)}(f(a)), & k = i, \\ \varphi_{k+n-1}^{(m+n-1)}(a), & k > i. \end{cases} * \sigma' \\
&\underset{\text{by (C2.2) and (C2.3)}}{=} \begin{cases} \varphi_{(\sigma')^{-1}(k)}^{(m+n-1)}(a), & k < i, \\ \sum\limits_{w=k}^{k+n-1} \varphi_{(\sigma')^{-1}(k)}^{(m+n-1)}(a) + \sum\limits_{k \leq k_1' < k_2' \leq k+n-1} \psi_{k_1' k_2'}^{(m+n-1)}(f(a)), & k = i, \\ \varphi_{(\sigma')^{-1}(k+n-1)}^{(m+n-1)}(a), & k > i. \end{cases} \\
&= \begin{cases} \varphi_k^{(m+n-1)}(a), & k < i, \\ \sum\limits_{w=k}^{k+n-1} \varphi_k^{(m+n-1)}(a) + \sum\limits_{k \leq k_1 < k_2 \leq k+n-1} \psi_{k_1 k_2}^{(m+n-1)}(f(a)), & k = i, \\ \varphi_{k+n-1}^{(m+n-1)}(a), & k > i. \end{cases}
\end{aligned}$$

Hence (E2.1.3) holds. It follows that $\mathcal{P}$ is an operad.

By (C3.1), we know that $\mathbb{1}_2$ is a 2-unit and $\mathcal{P}$ is 2-unitary operad. We define a morphism $u_{\mathcal{P}} : \mathcal{C}om \to \mathcal{P}$ by sending $\mathbb{1}_m \to \mathbb{1}_m$ for all $m \geq 0$. Note that $u_{\mathcal{P}}$ is an operadic morphism by (C2.1) and (C3.1). It follows that $\mathcal{P}$ is $\mathcal{C}om$-augmented.

Let $\{a_j\}_{j=1}^d$ be a basis of $\bar{A}$ and $\{\mu_k\}_{k=1}^m$ a basis of M where d is the dimension of $\bar{A}$ and m is the dimension of M. By construction,

$$\{\mathbb{1}_n,\quad \varphi_i^{(n)}(a_j) := \mathbb{1}_n \underset{i}{\circ} a_j,\quad \psi_{i_1 i_2}^{(n)}(\mu_k) := (\mathbb{1}_{n-1} \underset{1}{\circ} \mu_k) * c_{i_1 i_2}$$
$$\mid i \in [n], j \in [d], k \in [m], 1 \leq i_1 < i_2 \leq n\}$$

is a $\Bbbk$-basis of $\mathcal{P}(n)$. As a consequence, the generating function of $\mathcal{P}$ is

$$G_{\mathcal{P}}(t) = \sum_{n=0}^{\infty}(1 + dn + m\frac{n(n-1)}{2})t^n = \frac{1}{1-t} + d\frac{t}{(1-t)^2} + m\frac{t^2}{(1-t)^3}.$$

Therefore $\mathcal{P}$ has GK-dimension ≤ 3. It is clear that $\operatorname{GKdim}\mathcal{P} \leq 2$ if and only if $m = 0$ if and only if $M = 0$. □

EXAMPLE 5.2. Let A be an augmented algebra with the augmentation ideal $\bar{A}$, and let $\mathcal{P}$ be the corresponding operad of the trident algebra $(A, \operatorname{Ker}\pi, E)$ as in Example 4.8. Then an algebra Λ over $\mathcal{P}$ is a commutative associative algebra endowed with operators $\delta\colon \Lambda \to \Lambda$ (for all $\delta \in \bar{A}$) satisfying

$$\delta(xy)\delta'(z) = x\delta(y)\delta'(z) + y\delta(x)\delta'(z),$$

and

$$\delta(x)\delta'(y)\delta''(z) = 0$$

for all $\delta, \delta', \delta'' \in \bar{A}$ and $x, y, z \in \Lambda$.

Let $\mathcal{P}'$ be the operad corresponding to the second trident algebra $(A, \operatorname{Ker}\pi', E')$ as in Example 4.8. Then an algebra Λ over $\mathcal{P}'$ is a commutative associative algebra endowed with derivations $\delta\colon \Lambda \to \Lambda$ (for all $\delta \in \bar{A}$) satisfying

$$\delta(x)\delta'(y) + \delta'(x)\delta(y) = 0$$

and

$$\delta(x)\delta'(y)\delta''(z) = 0$$

for all $\delta, \delta', \delta'' \in \bar{A}$ and $x, y, z \in \Lambda$.

5.2. Classification of 2-unitary operads of GK-dimension 3. Now we are ready to prove the main theorem.

THEOREM 5.3. *The category $\mathcal{C}$ of trident algebras (A, M, f, g) is equivalent to the category $\mathcal{D}$ of $\mathcal{C}om$-augmented operads of GK-dimension ≤ 3.*

PROOF. Define a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ as follows:

(i) For any trident algebra (A, M, f, g),

$$\mathcal{F}(A, M, f, g) := F(A, M, f, g)$$

where $F(A, M, f, g)$ is given in Theorem 5.1, namely,

$$\mathcal{F}(A, M, f, g) = \{\mathcal{P}(n)\}_{n\geq 0} = \{\Bbbk \mathbb{1}_n \oplus \bigoplus_{k=1}^{n} \bar{A}_k^{(n)} \oplus \bigoplus_{1\leq i<j\leq n} M_{ij}^{(n)}\}_{n\geq 0}.$$

By the main theorem [Theorem 5.1], $\mathcal{F}(A, M, f, g)$ is a $\mathcal{C}om$-augmented operad of GK-dimension ≤ 3.

(ii) For a morphism $(\alpha, \beta)\colon (A, M, f, g) \to (A', M', f', g')$, we define an operadic morphism

$$\Phi = \mathcal{F}(\alpha, \beta)\colon \{\mathcal{P}(n)\} \to \{\mathcal{P}'(n)\}$$

as follows:

$$\begin{aligned} \Phi_n(\mathbb{1}_n)\colon &= \mathbb{1}'_n; \\ \Phi_n(\varphi_k^n(a))\colon &= {\varphi'}_k^n(\alpha(a)), \quad \text{for } a \in \bar{A}; \\ \Phi_n(\psi_{ij}^n(\mu))\colon &= {\psi'}_{ij}^n(\beta(\mu)), \quad \text{for } \mu \in M. \end{aligned}$$

By a direct calculation, it follows easily from (C2.1)–(C2.3) and (C3.1)–(C3.9) that Φ is a morphism of operads since (α, β) is a morphism in the category $\mathcal{C}$.

Conversely, we define a functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ as follows: for an object $\mathcal{P}$ in category $\mathcal{D}$, we define

$$\mathcal{G}(\mathcal{P}) = (A, M, f, g)$$

where $A = \mathcal{P}(1)$, $M = {}^2\Upsilon_{\mathcal{P}}(2)$, and

$$f: \quad {}^1\Upsilon(1) \to {}^2\Upsilon(2), \qquad a \mapsto a \circ \mathbb{1}_2 - \mathbb{1}_2 \underset{1}{\circ} a - \mathbb{1}_2 \underset{2}{\circ} a;$$
$$g: \quad {}^1\Upsilon(1) \otimes {}^1\Upsilon(1) \to {}^2\Upsilon(2), \qquad (a, b) \mapsto (\mathbb{1}_2 \underset{1}{\circ} a) \underset{2}{\circ} b.$$

We show next that (A, M, f, g) is a trident algebra. By definitions, $A := \mathcal{P}(1)$ is an associative $\Bbbk$-algebra with identity $\mathbb{1}_1$. Considering the map $\pi^{\varnothing} : \mathcal{P}(1) \to \mathcal{P}(0) = \Bbbk\mathbb{1}_0, \theta \mapsto \theta \circ \mathbb{1}_0$, we know that A is an augmented algebra with the augmentation ideal $\operatorname{Ker} \pi^{\varnothing} = {}^1\Upsilon_{\mathcal{P}}(1)$. By the definition of truncation ideals, $M := {}^2\Upsilon_{\mathcal{P}}(2)$ is a $\Bbbk\mathbb{S}_2$-submodule of $\mathcal{P}(2)$, and is an $(A, A \otimes A)$-bimodule with the module actions given by the related composition map. Since $\mathcal{P}$ is a $\mathcal{C}om$-augmented (hence 2-unitary) operad of GK-dimension ≤ 3, we have

$$(a \circ \mathbb{1}_2 - \mathbb{1}_2 \underset{1}{\circ} a - \mathbb{1}_2 \underset{2}{\circ} a) \underset{i}{\circ} \mathbb{1}_0 = 0,$$

for all $a \in \bar{A}$ and $i = 1, 2$, and

$$((\mathbb{1}_2 \underset{1}{\circ} a) \underset{2}{\circ} b) \underset{i}{\circ} \mathbb{1}_0 = 0,$$

for all $a, b \in \bar{A}$ and $i = 1, 2$. Therefore $a \circ \mathbb{1}_2 - \mathbb{1}_2 \underset{1}{\circ} a - \mathbb{1}_2 \underset{2}{\circ} a$ and $(\mathbb{1}_2 \underset{1}{\circ} a) \underset{2}{\circ} b$ are in M. Therefore f maps from $\bar{A} \to M$ and g maps from $\bar{A}^{\otimes 2} \to M$.

For $a, b \in A$ and $\mu \in M$, let $a \cdot \mu$ be $a \circ \mu$ and $\mu \cdot (a \otimes b)$ be $\mu \circ (a, b)$ both of which are in M. Then, for all $a \in A$, $\mu \in M$,

$$a \cdot (\mu * (2,1)) = a \underset{1}{\circ} (\mu * (2,1)) = (a \underset{1}{\circ} \mu) * (\mathbb{1}_1 \underset{1}{\circ} (2,1)) = (a \cdot \mu) * (2,1)$$

which shows that (E4.1.1) holds. For $a, b \in A$, we have

$$\begin{aligned}(\mu * (2,1)) \cdot (a \otimes b) =& (\mu * (2,1)) \underset{1}{\circ} a \underset{2}{\circ} b = ((\mu \underset{2}{\circ} a) * ((2,1) \underset{1}{\circ} \mathbb{1}_1)) \underset{2}{\circ} b \\ =& ((\mu \underset{2}{\circ} a) \underset{1}{\circ} b) * ((2,1) \underset{2}{\circ} \mathbb{1}_1)) = (\mu \cdot (b \otimes a)) * (2,1),\end{aligned}$$

which shows that (E4.1.2) holds. Hence M is a trident A-module.

For $\bar{a}, \bar{b} \in \bar{A}$,

$$\begin{aligned}f(\bar{a}\bar{b}) =& (\bar{a}\bar{b}) \circ \mathbb{1}_2 - \mathbb{1}_2 \underset{1}{\circ} (\bar{a}\bar{b}) - \mathbb{1}_2 \underset{2}{\circ} (\bar{a}\bar{b}) \\ =& \bar{a} \circ (\bar{b} \circ \mathbb{1}_2 - \mathbb{1}_2 \underset{1}{\circ} \bar{b} - \mathbb{1}_2 \underset{2}{\circ} \bar{b}) + (\bar{a} \circ \mathbb{1}_2 - \mathbb{1}_2 \underset{1}{\circ} \bar{a} - \mathbb{1}_2 \underset{2}{\circ} \bar{a}) \underset{1}{\circ} \bar{b} \\ &+ (\bar{a} \circ \mathbb{1}_2 - \mathbb{1}_2 \underset{1}{\circ} \bar{a} - \mathbb{1}_2 \underset{2}{\circ} \bar{a}) \underset{2}{\circ} \bar{b} + (\mathbb{1}_2 \underset{1}{\circ} \bar{a}) \underset{2}{\circ} \bar{b} + (\mathbb{1}_2 \underset{1}{\circ} \bar{b}) \underset{2}{\circ} \bar{a} \\ =& \bar{a} f(\bar{b}) + f(\bar{a}) \underset{1}{\circ} \bar{b} + f(\bar{a}) \underset{2}{\circ} \bar{b} + g(\bar{a}, \bar{b}) + g(\bar{b}, \bar{a}).\end{aligned}$$

Hence (E4.4.1) holds. For $\bar{a}, \bar{b}, \bar{c} \in \bar{A}$,

$$\begin{aligned}f(\bar{a}) \cdot (\bar{b}, \bar{c}) =& ((\bar{a} \circ \mathbb{1}_2 - \mathbb{1}_2 \underset{1}{\circ} \bar{a} - \mathbb{1}_2 \underset{2}{\circ} \bar{a}) \underset{1}{\circ} \bar{b}) \underset{2}{\circ} \bar{c} \\ =& \bar{a}((\mathbb{1}_2 \underset{1}{\circ} \bar{b}) \underset{2}{\circ} \bar{c}) - (\mathbb{1}_2 \underset{1}{\circ} (\bar{a}\bar{b})) \underset{2}{\circ} \bar{c} - (\mathbb{1}_2 \underset{1}{\circ} \bar{b}) \underset{2}{\circ} (\bar{a}\bar{c}) \\ =& \bar{a} g(\bar{b}, \bar{c}) - g(\bar{a}\bar{b}, \bar{c}) - g(\bar{b}, \bar{a}\bar{c}).\end{aligned}$$

Hence (E4.4.2) holds. Therefore (A, M, f, g) is a trident algebra. It follows that $\mathcal{G}(\mathcal{P})$ is an object in the category $\mathcal{C}$.

For the operad morphism $\Psi\colon \mathcal{P} \to \mathcal{P}'$, we define the morphism

$$(\alpha, \beta) = \mathcal{G}(\Psi)\colon (\mathcal{P}(1), {}^2\Upsilon_{\mathcal{P}}(2), f, g) \to (\mathcal{P}'(1), {}^2\Upsilon'_{\mathcal{P}'}(2), f', g')$$

as follows:

$$\mathcal{G}(\Psi) := (\Psi(1), \Psi(2)|_{{}^2\Upsilon(2)}).$$

It is routine to check that $\mathcal{G}$ is a functor from $\mathcal{D} \to \mathcal{C}$.

Finally, it is clear from the definition that $\mathcal{G}\mathcal{F}$ is the identity functor and it follows from Lemma 3.4 that $\mathcal{F}\mathcal{G}$ is naturally isomorphic to the identity functor. The assertion follows. □

6. Connected 2-unitary operads of GK-dimension 3

In this short section, we focus on connected 2-unitary operads of GK-dimension 3. Recall that a connected operad means a unitary operad with $\mathcal{P}(1) = \Bbbk\mathbb{1}$ [Definition 2.5(1)]. Let $\mathcal{P}$ be a connected 2-unitary operad. By Theorems 5.1 and 5.3, it is easily seen that the corresponding trident algebra of $\mathcal{P}$ is of the form $(\Bbbk, M, 0, 0)$, where M is a right $\mathbb{S}_2$-module. It follows that there is one-to-one correspondence betweeen connected 2-unitary operads $\mathcal{P}$ of GK-dimension 3 and nonzero right $\mathbb{S}_2$-modules M.

For the rest of this section we assume that char $\Bbbk \neq 2$. Suppose that

$$M = \Bbbk\tau_1 \oplus \cdots \oplus \Bbbk\tau_p \oplus \Bbbk\tau_{p+1} \oplus \cdots \Bbbk\tau_{p+q}$$

is an irreducible decomposition of the right $\mathbb{S}_2$-module M, where $\tau_i * (2,1) = -\tau_i$ for $1 \leq i \leq p$, $\tau_j * (2,1) = \tau_j$ for $p+1 \leq j \leq p+q$. Denote by $\mathcal{T}_{p,q}$ the connected 2-unitary operad corresponding to M. Then we have

$$\mathcal{T}_{p,q}(0) = \Bbbk\mathbb{1}_0, \quad \mathcal{T}_{p,q}(1) = \Bbbk\mathbb{1}_1,$$

$$\mathcal{T}_{p,q}(2) = \Bbbk\mathbb{1}_2 \oplus \Bbbk\tau_1 \oplus \cdots \oplus \Bbbk\tau_p \oplus \Bbbk\tau_{p+1} \oplus \cdots \oplus \Bbbk\tau_{p+q},$$

and for $n \geq 3$

$$\mathcal{T}_{p,q}(n) = \Bbbk\mathbb{1}_n \oplus \bigoplus_{1 \leq i_1 < i_2 \leq n, 1 \leq s \leq p+q} \Bbbk\tau^n_{(i_1,i_2),s}$$

as a $\Bbbk$-vector space.

The right action of $\mathbb{S}_n$ on $\mathcal{T}_{p,q}(n)$ is given by

(1) $\mathbb{1}_n * \sigma = \mathbb{1}_n$;

(2) $\tau^{(n)}_{(i_1,i_2),s} * \sigma = \begin{cases} \tau^n_{(\sigma^{-1}(i_1),\sigma^{-1}(i_2)),s}, & \sigma^{-1}(i_1) < \sigma^{-1}(i_2), 1 \leq s \leq p+q, \\ -\tau^n_{(\sigma^{-1}(i_2),\sigma^{-1}(i_1)),s}, & \sigma^{-1}(i_1) > \sigma^{-1}(i_2), 1 \leq s \leq p, \\ \tau^n_{(\sigma^{-1}(i_2),\sigma^{-1}(i_1)),s}, & \sigma^{-1}(i_1) > \sigma^{-1}(i_2), p+1 \leq s \leq p+q. \end{cases}$

for all $\sigma \in \mathbb{S}_n$.

The partial composition

$$\mathcal{T}_{p,q}(m) \underset{i}{\circ} \mathcal{T}_{p,q}(n) \to \mathcal{T}_{p,q}(m+n-1)$$

is given by

(i) $\mathbb{1}_m \underset{i}{\circ} \mathbb{1}_n = \mathbb{1}_{m+n-1}$ for all $1 \leq i \leq m$.

(ii) $\mathbb{1}_m \underset{i}{\circ} \tau^n_{(i_1 i_2),s} = \tau^{m+n-1}_{(i+i_1-1,i+i_2-1),s}$.

(iii) $\tau^m_{(i_1 i_2),s} \underset{i}{\circ} \mathbb{1}_n = \begin{cases} \sum_{k=i_1}^{i_1+n-1} \tau^{m+n-1}_{(k,i_2+n-1),s}, & i = i_1 < i_2, \\ \sum_{k=i_2}^{i_2+n-1} \tau^{m+n-1}_{(i_1,k),s}, & i_1 < i = i_2, \\ \tau^{m+n-1}_{(i_1,i_2+n-1),s}, & i_1 < i < i_2, \\ \tau^{m+n-1}_{(i_1 i_2),s}, & i_1 < i_2 < i, \\ \tau^{m+n-1}_{(i_1+n-1,i_2+n-1),s}, & i < i_1 < i_2. \end{cases}$

(iv) $\tau^m_{(i_1 i_2),s_1} \underset{i}{\circ} \tau^n_{(j_1 j_2),s_2} = 0$ for all $1 \le i \le m$.

As a consequence of Theorem 1.2 we have the following.

Corollary 6.1. *The category of connected 2-unitary operads of GK-dimension ≤ 3 is equivalent to the category of right $\mathbb{S}_2$-modules.*

Example 6.2. Let $M = \Bbbk\tau$ be a 1-dimensional vector space with right $\mathbb{S}_2$-action $\tau * (2,1) = -\tau$. Consider the trivial trident algebra $(\Bbbk, \Bbbk\tau, 0, 0)$. The correspond operad $\mathcal{P} = F(\Bbbk, \Bbbk\tau, 0, 0)$ in Theorem 5.1 is just $\mathcal{T}_{1,0}$, which is isomorphic to the quotient operad $\mathcal{P}ois/(\tau \underset{1}{\circ} \tau = 0)$. An algebra Λ over $\mathcal{P}$ is just a Poisson algebra with $\{\{x,y\},z\} = 0$ for any $x,y,z \in \Lambda$.

On the other hand, the quotient operad $\mathcal{A}ss/{}^3\varUpsilon_{\mathcal{A}ss}$ is also isomorphic to $\mathcal{T}_{1,0}$. Therefore we obtain an isomorphism of operads $\mathcal{P}ois/(\tau \underset{1}{\circ} \tau) \cong \mathcal{A}ss/{}^3\varUpsilon_{\mathcal{A}ss}$.

In fact, an algebra Λ over $\mathcal{A}ss/{}^3\varUpsilon_{\mathcal{A}ss}$ is an associative algebra satisfying $z(xy - yx) = (xy - yx)z$, or equivalently, $[[x,y],z] = 0$ for all $x,y,z \in \Lambda$, where $[x,y] = xy - yx$. Then we can define a new product $\star$ by $x \star y = \frac{xy+yx}{2}$ for all $x,y \in \Lambda$. It is direct to check that $(\Lambda, \star, [-,-])$ forms a Poisson algebra, and hence an algebra over $\mathcal{P}ois/(\tau \underset{1}{\circ} \tau)$ since $[[x,y],z] = 0$ for all $x,y,z \in \Lambda$.

Conversely, let Λ be a Poisson algebra with $\{\{x,y\},z\} = 0$ for any $x,y,z \in \Lambda$. We may define a new product $\circ$ on Λ by $x \circ y = \frac{xy+\{x,y\}}{2}$. Then $\{x,y\} = x \circ y - y \circ x$ for any $x,y \in \Lambda$ and $\circ$ satisfies the associativity law, which means that $(\Lambda, \circ)$ is an algebra over $\mathcal{A}ss/{}^3\varUpsilon_{\mathcal{A}ss}$.

It is also easy to see that $\mathcal{P}ois/{}^3\varUpsilon_{\mathcal{P}ois} \cong \mathcal{A}ss/{}^3\varUpsilon_{\mathcal{A}ss}$.

Similar to Example 6.2, for any 2-unitary operad $\mathcal{P}$, $\mathcal{P}/{}^3\varUpsilon_{\mathcal{P}}$ has GK-dimension ≤ 3. If further $\mathcal{P}$ is connected, then $\mathcal{P}/{}^3\varUpsilon_{\mathcal{P}}$ is a connected 2-unitary operad with GK-dimension ≤ 3.

Example 6.3. Let $M = \Bbbk\nu$ be a 1-dimensional vector space with right $\mathbb{S}_2$-action given by $\nu * (2,1) = \nu$. Then the corresponding operad $\mathcal{P} = F(\Bbbk, \Bbbk\nu, 0, 0)$ is just $\mathcal{T}_{0,1}$. An algebra over $\mathcal{P}$ is a commutative associative algebra Λ equipped with a binary operation $\nu\colon \Lambda \otimes \Lambda \to \Lambda$ satisfying

$$\begin{aligned} \nu(a,b) =& \nu(b,a), \\ \nu(ab,c) =& a\nu(b,c) + \nu(a,c)b, \\ \nu(\nu(a,b),c) =& 0, \end{aligned}$$

for all $a,b,c \in \Lambda$. By Corollary 6.1, $\mathcal{T}_{0,1} \ncong \mathcal{T}_{1,0}$.

7. Comments, examples, and remarks

We refer to [**LV**] for the definition of Hadamard product $-\underset{\mathrm{H}}{\otimes}-$ and a Hopf operad. Recall that the *Hadamard product* $\mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{Q}$ of the operads $\mathcal{P}$ and $\mathcal{Q}$ is defined to be

$$(\mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{Q})(n)=\mathcal{P}(n)\otimes\mathcal{Q}(n),$$

for all $n\geq 0$ with the partial composition

$$(\mu_1\otimes\nu_1)\underset{i}{\circ}(\mu_2\otimes\nu_2)=(\mu_1\underset{i}{\circ}\mu_2)\otimes(\nu_1\underset{i}{\circ}\nu_2),$$

for $\mu_1\otimes\nu_1\in(\mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{Q})(m)$, $\mu_2\otimes\nu_2\in(\mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{Q})(n)$, and $m\geq 1, n\geq 0, 1\leq i\leq m$. Clearly, the operad $\mathcal{C}om$ is obviously a unit for Hadamard product.

A *Hopf operad* is a symmetric operad $\mathcal{P}$ with a morphism of operads $\Delta\colon\mathcal{P}\to\mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{P}$ called the coproduct of $\mathcal{P}$ and a morphism of operads $\epsilon_{\mathcal{P}}\colon\mathcal{P}\to\mathcal{C}om$ called the counit, which is supposed to be coassociative and counital.

Definition 7.1. Let $\mathcal{P}$ be a Hopf operad. We say that $\mathcal{P}$ is a *$\mathcal{C}om$-augmented Hopf operad* if

(1) $\mathcal{P}$ is $\mathcal{C}om$-augmented and the composition

$$\mathcal{C}om\xrightarrow{u_{\mathcal{P}}}\mathcal{P}\xrightarrow{\epsilon}\mathcal{C}om$$

is the identity map, and

(2) the following diagram is commutative

$$\begin{tikzcd}
\mathcal{C}om \arrow[r, "\cong"] \arrow[d, "u_{\mathcal{P}}"'] & \mathcal{C}om\underset{\mathrm{H}}{\otimes}\mathcal{C}om \arrow[d, "u_{\mathcal{P}}\underset{\mathrm{H}}{\otimes}u_{\mathcal{P}}"] \\
\mathcal{P} \arrow[r, "\Delta"'] & \mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{P}.
\end{tikzcd}$$

Remark 7.2. A Hopf operad satisfying the condition (1) in Definition 7.1 is also called a unital augmented connected Hopf operad in [**Kh**, Definition 2.5].

Proposition 7.3. *Let $\mathcal{P}$ be a $\mathcal{C}om$-augmented Hopf operad of GK-dimension ≤ 2. Then $\mathcal{P}=\mathcal{C}om$.*

Proof. Note that $\Delta:\mathcal{P}\to\mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{P}$ is an operadic morphism. Write $\mathcal{P}$ as $F(A,0,0,0)$ as given by Theorem 1.2 (and as in Theorem 1.1).

Suppose $\bar{A}\neq 0$. Then $\operatorname{GKdim}\mathcal{P}=2$ and $\operatorname{GKdim}\mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{P}=3$. Since $\mathcal{P}\underset{\mathrm{H}}{\otimes}\mathcal{P}$ has GK-dimension 3, it is of the form $F(A\odot A)$ where $A\odot A$ is given in Example 4.9. By Theorem 5.3, Δ induces a morphism of trident algebras $(A,0,0,0)\to A\odot A:=(A\otimes A,M,f,g)$. Then $f\Delta\mid_{\mathcal{P}(1)}\colon A\to A\otimes A\to M$ is zero. We claim that $A=\Bbbk$. If not, let $0\neq a\in\bar{A}$ and write $\Delta\mid_{\mathcal{P}(1)}(a)=1\otimes a+a\otimes 1+\sum a_{(1)}\otimes a_{(2)}$ where $a_{(1)},a_{(2)}\in\bar{A}$. Then, by three equations in Example 4.9(1),

$$\begin{aligned}0=f\Delta(a)&=f(1\otimes a+a\otimes 1+\sum a_{(1)}\otimes a_{(2)})\\&=\sum((a_{(1)}\otimes 1_A)\otimes(1_A\otimes a_{(2)})+(1_A\otimes a_{(1)})\otimes(a_{(2)}\otimes 1_A)).\end{aligned}$$

Therefore $\sum a_{(1)}\otimes a_{(2)}=0$, and consequently, a is a primitive element. By Definition 7.1(2), $\mathbb{1}_n$ is group-like, i.e., $\Delta(\mathbb{1}_n)=\mathbb{1}_n\otimes\mathbb{1}_n$ for all n. Since each a is primitive, it follows from (C3.2) that each $\varphi_i^{(n)}(a)$ is primitive, i.e., $\Delta(\varphi_i^{(n)}(a)) = \varphi_i^{(n)}(a) \otimes \mathbb{1}_n + \mathbb{1}_n \otimes \varphi_i^{(n)}(a)$ for all i, n. Since $\mathcal{P}$ has GK-dimension ≤ 2, $\varphi_1^{(2)}(a) \underset{1}{\circ} \varphi_1^{(3)}(a) = 0$ by (C3.5). But

$$\begin{aligned}
&\Delta(\varphi_1^{(2)}(a) \underset{1}{\circ} \varphi_1^{(3)}(a)) \\
&\qquad =(\varphi_1^{(2)}(a) \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \varphi_1^{(2)}(a)) \underset{1}{\circ} (\varphi_1^{(3)}(a) \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes \varphi_1^{(3)}(a)) \\
&\qquad =(\varphi_1^{(2)}(a) \otimes \mathbb{1}_2) \underset{1}{\circ} (\mathbb{1}_3 \otimes \varphi_1^{(3)}(a)) + (\mathbb{1}_2 \otimes \varphi_1^{(2)}(a)) \underset{1}{\circ} (\varphi_1^{(3)}(a) \otimes \mathbb{1}_3) \\
&\qquad =(\sum_{k=1}^{2} \varphi_k^{(4)}(a)) \otimes \varphi_1^{(4)}(a) + \varphi_1^{(4)}(a) \otimes (\sum_{k=1}^{2} \varphi_k^{(4)}(a)) \\
&\qquad =\varphi_2^{(4)}(a) \otimes \varphi_1^{(4)}(a) + \varphi_1^{(4)}(a) \otimes \varphi_2^{(4)}(a) + 2\varphi_1^{(4)}(a) \otimes \varphi_1^{(4)}(a) \\
&\qquad \neq 0,
\end{aligned}$$

yielding a contradiction. Therefore $\bar{A} = 0$ and $\mathcal{P} = \mathcal{C}om$. $\square$

The proof of the above proposition uses both Theorems 1.1 and 1.2, as well as the structure maps of trident algebras. We do know an easy proof without using trident algebras.

Unlike in the GK-dimension 2 case, the 2-unit of an operad needs not be unique.

Example 7.4. Let $\mathcal{P} = F(A, M, f, g)$ where $M \neq 0$. Let $\mathbb{1}_2$ be the canonical 2-unit of $\mathcal{P}$ given in the construction of $F(A, M, f, g)$. Let $\mathbb{1}_2' = \mathbb{1}_2 + \psi_{12}^{(2)}(m)$. It is easy to check that $\mathbb{1}_2'$ is a $2a$-unit [Lemma 3.2]. Suppose that $m * (2, 1) \neq m$. Then $(\mathcal{P}, \mathbb{1}_0, \mathbb{1}_1, \mathbb{1}_2')$ is a 2-unitary operad, but not $\mathcal{C}om$-augmented. As a consequence, we can not replace "$\mathcal{C}om$-augmented" by "2-unitary" in Theorem 1.2.

Remark 7.5. For non-2-unitary operads, we have the following remarks.

1. (1) By [**QXZZ**, Construction 7.1] there are a lot of symmetric operads of GK-dimension 3 that are not 2-unitary.
2. (2) In [**QXZZ**], an analog of Bergman's gap theorem of nonsymmetric operads is proved, namely, no finitely generated locally finite nonsymmetric operad has GK-dimension strictly between 1 and 2. In [**LQXZZZ**] the authors proved that there is no finitely generated locally finite symmetric operad with GK-dimension strictly between 1 and 2.
3. (3) It is an open question if there are finitely generated locally finite symmetric operads with GK-dimension strictly between 2 and 3, see [**QXZZ**, Question 0.8].
4. (4) For every $r \in \{0\} \cup \{1\} \cup [2, \infty)$ or $r = \infty$, the authors in [**QXZZ**] constructed an explicit non-symmetric operad of GK-dimension r.

The following lemma was proved in [**BYZ**, Theorem 6.5].

Lemma 7.6. *Let $\mathcal{P}$ be a 2-unitary operad of finite GK-dimension ≥ 3. Then $\mathcal{P}$ is not semiprime.*

Proof. If $\mathcal{P}$ is semiprime, by the proof of [**BYZ**, Theorem 6.5], ${}^2\Upsilon = 0$. So $\operatorname{GKdim} \mathcal{P} \leq 2$, yielding a contradiction. $\square$

The following example shows that $\underset{\mathrm{H}}{\otimes}$ does not preserve primeness.

EXAMPLE 7.7. Let $A = \Bbbk \oplus M_2(\Bbbk)$, considered as an augmented algebra. Then $\mathcal{P} := F(A, 0, 0, 0)$ is prime of GK-dimension 2 which can be verified by using structure Theorem 1.1. Since $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{P}$ has GK-dimension 3, it is not semiprime by Lemma 7.6.

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References

[BV] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973. MR420609

[BYZ] Y.-H. Bao, Y. Ye, and J. J. Zhang, *Truncation of unitary operads*, Adv. Math. **372** (2020), 107290, 75, DOI 10.1016/j.aim.2020.107290. MR4125516

[Dot] V. Dotsenko, *A Quillen adjunction between algebras and operads, Koszul duality, and the Lagrange inversion formula*, Int. Math. Res. Not. IMRN **14** (2019), 4281–4301, DOI 10.1093/imrn/rnx257. MR4023760

[DK] V. Dotsenko and A. Khoroshkin, *Gröbner bases for operads*, Duke Math. J. **153** (2010), no. 2, 363–396, DOI 10.1215/00127094-2010-026. MR2667136

[DMR] V. Dotsenko, M. Markl, and E. Remm, *Veronese powers of operads and pure homotopy algebras*, Eur. J. Math. **6** (2020), no. 3, 829–863, DOI 10.1007/s40879-019-00351-6. MR4151721

[DT] V. Dotsenko and P. Tamaroff *Tangent complexes and the Diamond Lemma*, Preprint, `arXiv:2010.14792`, (2020).

[Fi] D. R. Finston, *Polynomial algebras have polynomial growth*, Trans. Amer. Math. Soc. **300** (1987), no. 2, 535–556, DOI 10.2307/2000356. MR876465

[Fr1] B. Fresse, *Homotopy of operads and Grothendieck-Teichmüller groups. Part 1*, Mathematical Surveys and Monographs, vol. 217, American Mathematical Society, Providence, RI, 2017. The algebraic theory and its topological background, DOI 10.1090/surv/217.1. MR3643404

[Fr2] B. Fresse, *Homotopy of operads and Grothendieck-Teichmüller groups. Part 2*, Mathematical Surveys and Monographs, vol. 217, American Mathematical Society, Providence, RI, 2017. The applications of (rational) homotopy theory methods, DOI 10.1090/surv/217.2. MR3616816

[Kh] A. Khoroshkin, *Quadratic algebras arising from Hopf operads generated by a single element*, Lett. Math. Phys. **110** (2020), no. 8, 2053–2082, DOI 10.1007/s11005-020-01283-z. MR4126873

[KP] A. Khoroshkin and D. Piontkovski, *On generating series of finitely presented operads*, J. Algebra **426** (2015), 377–429, DOI 10.1016/j.jalgebra.2014.12.012. MR3301915

[KL] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Revised edition, Graduate Studies in Mathematics, vol. 22, American Mathematical Society, Providence, RI, 2000, DOI 10.1090/gsm/022. MR1721834

[LQXZZZ] Y. Li, Z.-H. Qi, Y.-J. Xu, J. J. Zhang, Z.-R. Zhang, and X.-G. Zhao, *Operads of Gelfand–Kirillov dimension one*, in preparation.

[LV] J.-L. Loday and B. Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012, DOI 10.1007/978-3-642-30362-3. MR2954392

[Lo] J.-L. Loday, *Generalized bialgebras and triples of operads* (English, with English and French summaries), Astérisque **320** (2008), x+116. MR2504663

[MSS] M. Markl, S. Shnider, and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002, DOI 10.1090/surv/096. MR1898414

[Ma] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 271, Springer-Verlag, Berlin-New York, 1972. MR420610

[QXZZ] Z. Qi, Y. Xu, J. J. Zhang, and X. Zhao, *Growth of nonsymmetric operads*, Indiana Univ. Math. J. **72** (2023), no. 2, 651–697. MR4596188

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